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Robotics as a Future and Emerging Technology

Akash K Singh, PhD
IBM Corporation Sacramento, USA

Abstract

Real-time in-vivo forward-viewing optical coherence tomography imaging has been demonstrated with a novel lens scanning based MEMS endoscope catheter. An endoscopic catheter with an outer dimension of 7 mm x 7 mm has been designed, manufactured and assembled. By employing high-speed spectral domain optical coherence tomography, in-vivo two-dimensional cross-sectional images of human skin tissues were obtained as a preliminary study. Imaging speed of 122 frames per second and axial resolution of 7.7 µm are accomplished. The operation voltages are only DC 3 V and AC 6 Vpp at resonance frequency of 122 Hz. The catheter can provide many opportunities for clinical applications such as compact packaging, long working distance and body safe low operating voltages.

***Keywords*-Biorobotics, neuro-robotics, cybernetic hand, bioinspired sensors, underactuated hand, biomimetic robot.**

I. INTRODUCTION

The first machines in history were basically composed of mechanical parts, mechanisms, and some form of actuators supplied by an energy source. For a few centuries, this has been the paradigm for machines like windmills, textile frames, steam trains, and ships. What really changed in modern times was the incorporation of electronics into this basic scheme, allowing the integration of sensors and control, and the evolution of human-machine interfaces. This gave origin to mechatronics as the modern paradigm of machine design and the baseline for the development of robotics. This paradigm is today adopted in most products, from appliances to vehicles, aircrafts, robots, and biomedical devices. The new complete scheme, as depicted in Figure 1, can be seen in an analogy with biological systems, integrating a musculo- skeletal apparatus with a nervous system and a circulatory apparatus. When machine design takes inspiration from biology, as in this analogy, then it can be referred to as biomechatronics. Robotics, especially, is now following this direction, with a stronger emphasis on biorobotics and biomedical applications.

The evolution of the paradigm of modern biomechatronics and robotics can be seen in two main directions, standing as two extremities of a range of future biomechatronics systems:

◆increasing the performance and miniaturization of the hardware platform

◆ increasing the intelligence of the integrated system. Regarding the first direction, the current challenge is to develop sophisticated machines with a higher level of miniaturization and performance, as they can be inspired by insects. Towards the other extremity, there is research on intelligent and autonomous robots, like humanoids. At intermediate levels levels, we can envisage the development of machines with a good degree of sophistication and performance and with a moderate degree of intelligence and that are more prone to human supervision and control or even to integration with natural bodies as bionic components. In this article, three projects funded by the European Commission in the 5th Framework Programme, in the Information Society Technology-Future and Emerging Technologies (IST-FET) program, are presented as implementing three levels of this evolution of the biomechatronic paradigm: from the biomimetic wormlike microrobot for endoscopic exploration to a cybernetic hand prosthesis to an anthropomorphic robotic platform implementing learning schemes for sensory motor coordination in manipulation.

II. THE IST-FET BIOMIMETIC STRUCTURES FOR LOCOMOTION IN THE HUMAN BODY PROJECT

The objectives of the biomimetic structures for locomotion in the human body (BIOLOCH) project are to understand motion and perception systems of lower animal forms, such as parasites, worms, insects, and even snakes and eels, and to design, model, and fabricate bioinspired mini- and micromachines able to navigate in tortuous and hostile cavities and, in particular, in the human body. This idea comes from the medical need to make more powerful tools for microendoscopy, which is one of the most challenging frontiers of modern medicine. The most significant objectives that represent parallel and complementary results are:

◆ the increase of knowledge in the field of biomechanics and neural control

◆ the definition of biologically inspired design paradigms
◆ the development and introduction of innovative hybrid manufacturing technologies.

The first step in the project is the study of the locomotion mechanisms used by animals that move in a wet environment containing large amounts of solid and semisolid materials. In order to understand better how to gain net advancement on this soft tissue, the mechanisms of attachment used by parasites, both internal and external, have been investigated. The interaction between a biomimetic worm and the gut wall has been analyzed, and objective parameters, like fracture and tear resistance of such tissue, have been estimated. The second step of the project consists of the development of a biomimetic control module. The unit action-perception reaction of insects, worms, and parasites has been studied in order to define a strategy of replication and implementation in an artificial, bioinspired, adaptable system. Finally, the third step consists of the biomechatronic design of the previously identified systems and of their implementation by innovative technologies integrating and merging different functions. In this article, we illustrate the selection of the most promising locomotion architectures, and we present some preliminary platforms, developed both for mimicking the selected models and for creating a better understanding of the interaction between the platforms and the locomotion environment.

III. BIOMIMETIC UNDULATORY MECHANISMS

The taxonomy of biological propulsion systems has been performed by dividing them between adhesion (or differential friction) systems and locomotion systems. This classification has been pursued in order to analyze the problem of locomotion in the human cavities from an engineering point of view. In fact, even the simplest biological creatures exploit quite complex and sophisticated motion mechanisms, which are hard to understand and replicate without a sort of a priori systematization. This approach has been implemented extensively by studying locomotion issues in the human colon for more application driven robotic projects [1]. These studies have demonstrated that the peculiar anatomy, tribology [2], and mechanics of the human gut and of other human cavities pose many problems in terms of adhesion and stopping rather than in terms of propulsion, which is often a physiological process (in the intestine, blood vessels, and other such organs). In this context, locomotion often indicates the displacement of adhesive or high friction contact points. On the basis of this analysis, the authors studied, designed, and implemented some solutions for the realization of bioinspired mechanisms for friction enhancement. Adhesion modules can possess very different configurations and structures; they can consist of extremely tiny structures interacting by atomic force with the substrate [3]; they can be based on bio-glues, as used in surgery; or they can be based on miniand microstructures, which generate high friction in one direction and low friction in the opposite direction, thus giving a preferred direction to the motion [4]. Concerning the displacement of adhesive points (i.e., the propulsion), the undulatory locomotion has been considered the most appropriate for locomoting robotic devices in solid, semisolid, and dirty environments, after an overall analysis of biological locomotion mechanisms (e.g., legged or polypedal locomotion and serpentine locomotion). Undulatory locomotion has been modeled for two different cases: waves longitudinal and waves transversal, regarding the motion direction (Figure 2) [5], [6]. Locomotion with longitudinal waves (i.e., peristaltic locomotion) consists of waves that propel in parallel with the direction of motion. This motion is exploited by earthworms (oligochaeta) and leeches. The difference between the earthworm locomotion and the leech locomotion (also called inchworm locomotion) is that the first consists of continuous waves and the second of discrete waves. Peristaltic locomotion is generated by the alternation of longitudinal and circular muscle contraction waves flowing from the head to the tail. The sites of longitudinal contraction are the anchor points; body extension is by circular contraction. The pattern of movement is initiated by anchoring the anterior end. As the longitudinal contraction wave moves posteriorly, it is slowly replaced by the circular contraction wave. The anterior end slowly and forcefully elongates, driving the tip farther over the surface. The tip then begins to dilate and anchor the anterior end as another longitudinal contraction wave develops. This sequence is repeated, and the worm moves forward. Reversing the direction of the contraction waves enables the worm to back up. Locomotion with transversal waves consists of waves that are transversal to the motion direction, and it is typical of paddleworms (errant and sedentary polychaete). The large range of locomotion modes employed by errant polychaete is related to the diversity and structural complexity of their habitat environment. One species that is generalized locomotion-wise and has been extensively studied in the literature is nereis diversicolor, a common intertidal polychaete. It inhabits muddy substrata, and it possesses the ability to efficiently burrow. Its motion is very active, and it is also able to swim. Its body can reach 60–120 mm and consists of approximately 200 segments, each bearing a pair of parapodia. The authors have developed several locomotion prototypes of oligochaeta and polychaete with the aim of studying the interaction of these platforms with different environments [7] and of testing the computational models of locomotion and evaluating the effect of friction on the motion [8]. With respect to pioneer activity on snakelike robots [9], the main feature of the current research consists of the ability to generate propulsion without exploiting wheels but understanding and exploiting the same mechanisms used by worms (e.g., legs, parapodia, and differential friction surfaces). Figure 3 shows a four-module artificial oligochaeta with shape memory alloy (SMA) actuators. It consists of four silicone shells embedding SMA springs, which contract when activated. The robot possesses an external controller that activates each module sequentially. The robot locomotion is enabled by small metal legs that produce a differential friction, thus generating the net advancement of the device on flat surfaces, slippery surfaces (e.g., TEFLON), and sloped surfaces (Table 1). A five-module artificial polychaete, developed by exploiting the same principles used for the four-module worm, is illustrated in Figure 4. In this case, the traveling body wave is generated by the alternative contraction of two sets of SMA springs, which bend sequentially the left and the right modules of the polychaete. In this case, locomotion is obtained also without introducing differential friction structures (e.g., small hooked legs); the slight mounting asymmetries of the robot body contribute to generate a preferential direction of advancement (Table 2). Finally, a robotic polychaete endowed with traditional actuators (servos dc minimotors) has been developed for testing the computational models of locomotion and evaluating the effect of friction onto the motion. The robotic platform, whose joints are driven by a sinusoidal wave while moving on sand, and a typical computational model for forward locomotion are shown in Figure 5 [8].

We consider the following anycast field equations defined over an open bounded piece of network and /or feature space $\Omega \subset R^d$. They describe the dynamics of the mean anycast of each of p node populations.

$$\begin{cases} (\frac{d}{dt}+l_i)V_i(t,r)=\sum_{j=1}^{p}\int_{\Omega} J_{ij}(r,\bar{r})S[(V_j(t-\tau_{ij}(r,\bar{r}),\bar{r})-h_{|j})]d\bar{r} \\ \qquad +I_i^{ext}(r,t), \qquad t\geq 0, 1\leq i\leq p, \\ V_i(t,r)=\phi_i(t,r) \qquad t\in[-T,0] \end{cases} \tag{1}$$

We give an interpretation of the various parameters and functions that appear in (1), Ω is finite piece of nodes and/or feature space and is represented as an open bounded set of R^d . The vector r and $\bar{r}$ represent points in Ω . The function $S: R \to (0,1)$ is the normalized sigmoid function:

$$S(z)=\frac{1}{1+e^{-z}} \tag{2}$$

It describes the relation between the input rate v_i of population i as a function of the packets potential, for example, $V_i = v_i = S[\sigma_i(V_i - h_i)]$. We note V the $p-$ dimensional vector $(V_1,\ldots,V_p)$. The p function $\phi_i, i=1,\ldots,p,$ represent the initial conditions, see below. We note ϕ the $p-$ dimensional vector $(\phi_1,\ldots,\phi_p)$. The p function $I_i^{ext}, i=1,\ldots,p,$ represent external factors from other network areas. We note I^{ext} the $p-$ dimensional vector $(I_1^{ext},\ldots,I_p^{ext})$. The $p\times p$ matrix of functions $J=\{J_{ij}\}_{i,j=1,\ldots,p}$ represents the connectivity between populations i and j, see below. The p real values $h_i, i=1,\ldots,p,$ determine the threshold of activity for each population, that is, the value of the nodes potential corresponding to 50% of the maximal activity. The p real positive values $\sigma_i, i=1,\ldots,p,$ determine the slopes of the sigmoids at the origin. Finally the p real positive values $l_i, i=1,\ldots,p,$ determine the speed at which each anycast node potential decreases exponentially toward its real value. We also introduce the function $S: R^p \to R^p$, defined by $S(x)=[S(\sigma_1(x_1-h_1)),\ldots,S(\sigma_p - h_p))],$ and the diagonal $p\times p$ matrix $L_0 = diag(l_1,\ldots,l_p)$. Is the intrinsic dynamics of the population given by the linear response of data transfer. $(\frac{d}{dt}+l_i)$ is replaced by $(\frac{d}{dt}+l_i)^2$ to use the alpha function response. We use $(\frac{d}{dt}+l_i)$ for simplicity although our analysis applies to more general intrinsic dynamics. For the sake, of generality, the propagation delays are not assumed to be identical for all populations, hence they are described by a matrix $\tau(r,\bar{r})$ whose element $\tau_{ij}(r,\bar{r})$ is the propagation delay between population j at $\bar{r}$ and population i at r. The reason for this assumption is that it is still unclear from anycast if propagation delays are independent of the populations. We assume for technical reasons that τ is continuous, that is $\tau \in C^0(\bar{\Omega}^2, R_+^{p\times p})$. Moreover packet data indicate that τ is not a symmetric function i.e., $\tau_{ij}(r,\bar{r}) \neq \tau_{ij}(\bar{r},r)$, thus no assumption is made about this symmetry unless otherwise stated. In order to compute the righthand side of (1), we need to know the node potential factor V on interval $[-T,0]$. The value of T is obtained by considering the maximal delay:

$$\tau_m = \max_{i,j(r,\bar{r}\in\Omega\times\Omega)} \tau_{i,j}(r,\bar{r}) \qquad (3)$$

Hence we choose $T=\tau_m$

A. Mathematical Framework

A convenient functional setting for the non-delayed packet field equations is to use the space $F=L^2(\Omega,R^p)$ which is a Hilbert space endowed with the usual inner product:

$$\langle V,U\rangle_F = \sum_{i=1}^{p}\int_\Omega V_i(r)U_i(r)dr \qquad (1)$$

To give a meaning to (1), we defined the history space $C=C^0([-\tau_m,0],F)$ with $\|\phi\|=\sup_{t\in[-\tau_m,0]}\|\phi(t)\|F,$ which is the Banach phase space associated with equation (3). Using the notation $V_t(\theta)=V(t+\theta),\theta\in[-\tau_m,0],$ we write (1) as

$$\begin{cases} \dot{V}(t) = -L_0V(t)+L_1S(V_t)+I^{ext}(t), \\ V_0=\phi\in C, \end{cases} \qquad (2)$$

Where

$$\begin{cases} L_1 : C\to F, \\ \phi\to\int_\Omega J(.,\bar{r})\phi(\bar{r},-\tau(.,\bar{r}))d\bar{r} \end{cases}$$

Is the linear continuous operator satisfying $\|L_1\|\le\|J\|_{L^2(\Omega^2,R^{p\times p})}$. Notice that most of the papers on this subject assume Ω infinite, hence requiring $\tau_m=\infty$.

Proposition 1.0 If the following assumptions are satisfied.

1. $J\in L^2(\Omega^2,R^{p\times p}),$
2. The external current $I^{ext}\in C^0(R,F),$
3. $\tau\in C^0(\overline{\Omega^2},R_+^{p\times p}),\sup_{\overline{\Omega^2}}\tau\le\tau_m.$

Then for any $\phi\in C,$ there exists a unique solution $V\in C^1([0,\infty),F)\cap C^0([-\tau_m,\infty,F)$ to (3)

Notice that this result gives existence on R_+, finite-time explosion is impossible for this delayed differential equation. Nevertheless, a particular solution could grow indefinitely, we now prove that this cannot happen.

B. Boundedness of Solutions

` A valid model of neural networks should only feature bounded packet node potentials.

Theorem 1.0 All the trajectories are ultimately bounded by the same constant R if $I\equiv\max_{t\in R^+}\|I^{ext}(t)\|_F<\infty.$

Proof :Let us defined $f:R\times C\to R^+$ as

$$f(t,V_t)\overset{def}{=}\langle -L_0V_t(0)+L_1S(V_t)+I^{ext}(t),V(t)\rangle_F=\frac{1}{2}\frac{d\|V\|_F^2}{dt}$$

We note $l=\min_{i=1,\ldots p} l_i$

$$f(t,V_t)\le -l\|V(t)\|_F^2+(\sqrt{p|\Omega|}\|J\|_F+I)\|V(t)\|_F$$

Thus, if

$$\|V(t)\|_F\ge 2\frac{\sqrt{p|\Omega|}.\|J\|_F+I}{l}\overset{def}{=}R,\ f(t,V_t)\le-\frac{lR^2}{2}\overset{def}{=}-\delta<0$$

Let us show that the open route of F of center 0 and radius $R, B_R,$ is stable under the dynamics of equation. We know that $V(t)$ is defined for all $t\ge 0s$ and that $f<0$ on ∂B_R, the boundary of B_R. We consider three cases for the initial condition V_0. If $\|V_0\|_C<R$ and set $T=\sup\{t\,|\,\forall s\in[0,t],V(s)\in\overline{B_R}\}.$ Suppose that $T\in R,$ then $V(T)$ is defined and belongs to $\overline{B_R}$, the closure of B_R, because $\overline{B_R}$ is closed, in effect to ∂B_R, we also have $\frac{d}{dt}\|V\|_F^2\,|_{t=T}=f(T,V_T)\le-\delta<0$ because $V(T)\in\partial B_R$. Thus we deduce that for $\varepsilon>0$ and small enough, $V(T+\varepsilon)\in\overline{B_R}$ which contradicts the definition of T. Thus $T\notin R$ and $\overline{B_R}$ is stable.

Because f<0 on $\partial B_R, V(0)\in\partial B_R$ implies that $\forall t>0,V(t)\in B_R$. Finally we consider the case $V(0)\in C\overline{B_R}$. Suppose that $\forall t>0,V(t)\notin\overline{B_R},$ then $\forall t>0,\frac{d}{dt}\|V\|_F^2\le-2\delta,$ thus $\|V(t)\|_F$ is monotonically decreasing and reaches the value of R in finite time when $V(t)$ reaches ∂B_R. This contradicts our assumption. Thus $\exists T>0\,|\,V(T)\in B_R$.

Proposition 1.1 : Let s and t be measured simple functions on X. for $E\varepsilon M$, define

$$\phi(E)=\int_E s\,d\mu \qquad (1)$$

Then ϕ is a measure on M.

$$\int_X (s+t)d\mu=\int_X s\,d\mu+\int_X td\mu \qquad (2)$$

Proof *:* If s and if $E_1, E_2, \ldots$ are disjoint members of M whose union is E, the countable additivity of μ shows that

$$\phi(E)=\sum_{i=1}^{n}\alpha_i\mu(A_i\cap E)=\sum_{i=1}^{n}\alpha_i\sum_{r=1}^{\infty}\mu(A_i\cap E_r)$$

$$=\sum_{r=1}^{\infty}\sum_{i=1}^{n}\alpha_i\mu(A_i\cap E_r)=\sum_{r=1}^{\infty}\phi(E_r)$$

Also, $\varphi(\phi)=0$, so that φ is not identically ∞.

Next, let s be as before, let $\beta_1,\ldots,\beta_m$ be the distinct values of t,and let $B_j=\{x:t(x)=\beta_j\}$ If $E_{ij}=A_i\cap B_j$, the

$$\int_{E_{ij}}(s+t)d\mu=(\alpha_i+\beta_j)\mu(E_{ij})$$

and $\int_{E_{ij}} sd\mu+\int_{E_{ij}} td\mu=\alpha_i\mu(E_{ij})+\beta_j\mu(E_{ij})$

Thus (2) holds with E_{ij} in place of X. Since X is the disjoint union of the sets E_{ij} $(1\le i\le n, 1\le j\le m)$, the first half of our proposition implies that (2) holds.

Theorem 1.1: If K is a compact set in the plane whose complement is connected, if f is a continuous complex function on K which is holomorphic in the interior of , and if $\varepsilon>0$, then there exists a polynomial P such that $\left|f(z)=P(z)\right|<\varepsilon$ for all $z\varepsilon K$. If the interior of K is empty, then part of the hypothesis is vacuously satisfied, and the conclusion holds for every $f\varepsilon C(K)$. Note that K need to be connected.

Proof: By Tietze's theorem, f can be extended to a continuous function in the plane, with compact support. We fix one such extension and denote it again by f. For any $\delta>0$, let $\omega(\delta)$ be the supremum of the numbers $\left|f(z_2)-f(z_1)\right|$ Where z_1 and z_2 are subject to the condition $\left|z_2-z_1\right|\le\delta$. Since f is uniformly continous, we have $\lim_{\delta\to 0}\omega(\delta)=0$ (1) From now on, δ will be fixed. We shall prove that there is a polynomial P such that

$$\left|f(z)-P(z)\right|<10{,}000\ \omega(\delta) \quad (z\varepsilon K) \qquad (2)$$

By (1), this proves the theorem. Our first objective is the construction of a function $\Phi\varepsilon C_c'(R^2)$, such that for all z

$$\left|f(z)-\Phi(z)\right|\le\omega(\delta), \qquad (3)$$

$$\left|(\partial\Phi)(z)\right|<\frac{2\omega(\delta)}{\delta}, \qquad (4)$$

And

$$\Phi(z)=-\frac{1}{\pi}\iint_X\frac{(\partial\Phi)(\zeta)}{\zeta-z}d\zeta\,d\eta \qquad (\zeta=\xi+i\eta), \qquad (5)$$

Where X is the set of all points in the support of Φ whose distance from the complement of K does not δ. (Thus X contains no point which is "far within" K.) We construct Φ as the convolution of f with a smoothing function A. Put $a(r)=0$ if $r>\delta$, put

$$a(r)=\frac{3}{\pi\delta^2}(1-\frac{r^2}{\delta^2})^2 \qquad (0\le r\le\delta), \qquad (6)$$

And define

$$A(z)=a(\left|z\right|) \qquad (7)$$

For all complex z. It is clear that $A\varepsilon C_c'(R^2)$. We claim that

$$\iint_{R^s}A=1, \qquad (8)$$

$$\iint_{R^2}\partial A=0, \qquad (9)$$

$$\iint_{R^3}\left|\partial A\right|=\frac{24}{15\delta}<\frac{2}{\delta}, \qquad (10)$$

The constants are so adjusted in (6) that (8) holds. (Compute the integral in polar coordinates), (9) holds simply because A has compact support. To compute (10), express ∂A in polar coordinates, and note that $\partial A/\partial\theta=0$,

$$\partial A/\partial r=-a',$$

Now define

$$\Phi(z)=\iint_{R^2}f(z-\zeta)Ad\xi d\eta=\iint_{R^2}A(z-\zeta)f(\zeta)d\xi d\eta \qquad (11)$$

Since f and A have compact support, so does Φ. Since

$$\Phi(z) - f(z)$$
$$= \iint_{R^2} [f(z-\zeta) - f(z)] A(\xi) d\xi d\eta \quad (12)$$

And $A(\zeta) = 0$ if $|\zeta| > \delta$, (3) follows from (8).

The difference quotients of A converge boundedly to the corresponding partial derivatives, since $A \varepsilon C_c^{'}(R^2)$. Hence the last expression in (11) may be differentiated under the integral sign, and we obtain

$$(\partial\Phi)(z) = \iint_{R^2} (\overline{\partial A})(z-\zeta) f(\zeta) d\xi d\eta$$
$$= \iint_{R^2} f(z-\zeta)(\partial A)(\zeta) d\xi d\eta$$
$$= \iint_{R^2} [f(z-\zeta) - f(z)](\partial A)(\zeta) d\xi d\eta \quad (13)$$

The last equality depends on (9). Now (10) and (13) give (4). If we write (13) with Φ_x and Φ_y in place of $\partial\Phi$, we see that Φ has continuous partial derivatives, if we can show that $\partial\Phi = 0$ in G, where G is the set of all $z \varepsilon K$ whose distance from the complement of K exceeds δ. We shall do this by showing that

$$\Phi(z) = f(z) \qquad (z \varepsilon G); \qquad (14)$$

Note that $\partial f = 0$ in G, since f is holomorphic there. Now if $z \varepsilon G$, then $z - \zeta$ is in the interior of K for all ζ with $|\zeta| < \delta$. The mean value property for harmonic functions therefore gives, by the first equation in (11),

$$\Phi(z) = \int_0^{\delta} a(r) r dr \int_0^{2\pi} f(z - re^{i\theta}) d\theta$$
$$= 2\pi f(z) \int_0^{\delta} a(r) r dr = f(z) \iint_{R^2} A = f(z) \qquad (15)$$

For all $z \ \varepsilon \ G$, we have now proved (3), (4), and (5) The definition of X shows that X is compact and that X can be covered by finitely many open discs $D_1, \ldots, D_n$, of radius 2δ, whose centers are not in K. Since $S^2 - K$ is connected, the center of each D_j can be joined to ∞ by a polygonal path in $S^2 - K$. It follows that each D_j contains a compact connected set E_j, of diameter at least 2δ, so that $S^2 - E_j$ is connected and so that $K \cap E_j = \phi$. with $r = 2\delta$. There are functions $g_j \varepsilon H(S^2 - E_j)$ and constants b_j so that the inequalities.

$$|Q_j(\zeta, z)| < \frac{50}{\delta}, \qquad (16)$$

$$\left| Q_j(\zeta, z) - \frac{1}{z-\zeta} \right| < \frac{4{,}000\delta^2}{|z-\zeta|^2} \qquad (17)$$

Hold for $z \notin E_j$ and $\zeta \in D_j$, if

$$Q_j(\zeta, z) = g_j(z) + (\zeta - b_j) g_j^2(z) \qquad (18)$$

Let Ω be the complement of $E_1 \cup \ldots \cup E_n$. Then Ω is an open set which contains K. Put $X_1 = X \cap D_1$ and $X_j = (X \cap D_j) - (X_1 \cup \ldots \cup X_{j-1})$, for $2 \le j \le n$,

Define

$$R(\zeta, z) = Q_j(\zeta, z) \qquad (\zeta \varepsilon X_j, z \varepsilon \Omega) \qquad (19)$$

And

$$F(z) = \frac{1}{\pi} \iint_X (\partial\Phi)(\zeta) R(\zeta, z) d\zeta d\eta \qquad (20)$$
$$(z \ \varepsilon \ \Omega)$$

Since,

$$F(z) = \sum_{j=1} \frac{1}{\pi} \iint_{X_i} (\partial\Phi)(\zeta) Q_j(\zeta, z) d\xi d\eta, \qquad (21)$$

(18) shows that F is a finite linear combination of the functions g_j and g_j^2. Hence $F \varepsilon H(\Omega)$. By (20), (4), and (5) we have

$$|F(z) - \Phi(z)| < \frac{2\omega(\delta)}{\pi\delta} \iint_X | R(\zeta, z)$$
$$- \frac{1}{z-\zeta} | d\xi d\eta \quad (z \ \varepsilon \ \Omega) \quad (22)$$

Observe that the inequalities (16) and (17) are valid with R in place of Q_j if $\zeta \ \varepsilon \ X$ and $z \ \varepsilon \ \Omega$. Now fix $z \ \varepsilon \ \Omega$., put $\zeta = z + \rho e^{i\theta}$, and estimate the integrand in (22) by (16) if $\rho < 4\delta$, by (17) if $4\delta \le \rho$. The integral in (22) is then seen to be less than the sum of

$$2\pi \int_0^{4\delta} \left(\frac{50}{\delta} + \frac{1}{\rho} \right) \rho d\rho = 808\pi\delta \qquad (23)$$

And

$$2\pi \int_{4\delta}^{\infty} \frac{4{,}000\delta^2}{\rho^2} \rho d\rho = 2{,}000\pi\delta. \qquad (24)$$

Hence (22) yields

$$|F(z) - \Phi(z)| < 6{,}000\omega(\delta) \qquad (z \ \varepsilon \ \Omega) \qquad (25)$$

Since $F \ \varepsilon \ H(\Omega)$, $K \subset \Omega$, and $S^2 - K$ is connected, Runge's theorem shows that F can be uniformly approximated on K by polynomials. Hence (3) and (25) show that (2) can be satisfied. This completes the proof.

Lemma 1.0 : Suppose $f \varepsilon C_c^{'}(R^2)$, the space of all continuously differentiable functions in the plane, with compact support. Put

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \qquad (1)$$

Then the following "Cauchy formula" holds:

$$f(z) = -\frac{1}{\pi}\iint_{R^2} \frac{(\partial f)(\zeta)}{\zeta - z} d\xi d\eta$$

$$(\zeta = \xi + i\eta) \qquad (2)$$

Proof: This may be deduced from Green's theorem. However, here is a simple direct proof:

Put $\varphi(r,\theta) = f(z + re^{i\theta}),\ r > 0,\ \theta$ real

If $\zeta = z + re^{i\theta}$, the chain rule gives

$$(\partial f)(\zeta) = \frac{1}{2}e^{i\theta}\left[\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right]\varphi(r,\theta) \qquad (3)$$

The right side of (2) is therefore equal to the limit, as $\varepsilon \to 0$, of

$$-\frac{1}{2}\int_{\varepsilon}^{\infty}\int_{0}^{2\pi}\left(\frac{\partial \varphi}{\partial r} + \frac{i}{r}\frac{\partial \varphi}{\partial \theta}\right) d\theta dr \qquad (4)$$

For each $r > 0, \varphi$ is periodic in θ, with period 2π. The integral of $\partial\varphi / \partial\theta$ is therefore 0, and (4) becomes

$$-\frac{1}{2\pi}\int_0^{2\pi} d\theta \int_{\varepsilon}^{\infty}\frac{\partial \varphi}{\partial r} dr = \frac{1}{2\pi}\int_0^{2\pi}\varphi(\varepsilon,\theta) d\theta \qquad (5)$$

As $\varepsilon \to 0,\ \varphi(\varepsilon,\theta) \to f(z)$ uniformly. This gives (2)

If $X^{\alpha} \in a$ and $X^{\beta} \in k[X_1,...X_n]$, then $X^{\alpha}X^{\beta} = X^{\alpha+\beta} \in a$, and so A satisfies the condition $(*)$. Conversely,

$$(\sum_{\alpha \in A} c_{\alpha}X^{\alpha})(\sum_{\beta \in \mathbb{N}^n} d_{\beta}X^{\beta}) = \sum_{\alpha,\beta} c_{\alpha}d_{\beta}X^{\alpha+\beta} \qquad (\textit{finite sums}),$$

and so if A satisfies $(*)$, then the subspace generated by the monomials $X^{\alpha}, \alpha \in a$, is an ideal. The proposition gives a classification of the monomial ideals in $k[X_1,...X_n]$: they are in one to one correspondence with the subsets A of $\mathbb{N}^n$ satisfying $(*)$. For example, the monomial ideals in $k[X]$ are exactly the ideals $(X^n),\ n \geq 1$, and the zero ideal (corresponding to the empty set A). We write $\langle X^{\alpha} \mid \alpha \in A\rangle$ for the ideal corresponding to A (subspace generated by the $X^{\alpha}, \alpha \in a$).

LEMMA 1.1. Let S be a subset of $\mathbb{N}^n$. The the ideal a generated by $X^{\alpha}, \alpha \in S$ is the monomial ideal corresponding to

$$A \overset{df}{=} \{\beta \in \mathbb{N}^n \mid \beta - \alpha \in \mathbb{N}^n, \quad some\ \alpha \in S\}$$

Thus, a monomial is in a if and only if it is divisible by one of the $X^{\alpha}, \alpha \in| S$

PROOF. Clearly A satisfies $(*)$, and $a \subset \langle X^{\beta} \mid \beta \in A\rangle$. Conversely, if $\beta \in A$, then $\beta - \alpha \in \mathbb{N}^n$ for some $\alpha \in S$, and $X^{\beta} = X^{\alpha}X^{\beta-\alpha} \in a$. The last statement follows from the fact that $X^{\alpha} \mid X^{\beta} \Leftrightarrow \beta - \alpha \in \mathbb{N}^n$. Let $A \subset \mathbb{N}^n$ satisfy $(*)$. From the geometry of A, it is clear that there is a finite set of elements $S = \{\alpha_1,...\alpha_s\}$ of A such that $A = \{\beta \in \mathbb{N}^n \mid \beta - \alpha_i \in \mathbb{N}^2,\ some\ \alpha_i \in S\}$ (The α_i's are the corners of A) Moreover, $a \overset{df}{=} \langle X^{\alpha} \mid \alpha \in A\rangle$ is generated by the monomials $X^{\alpha_i}, \alpha_i \in S$.

DEFINITION 1.0. For a nonzero ideal a in $k[X_1,...,X_n]$, we let $(LT(a))$ be the ideal generated by

$$\{LT(f) \mid f \in a\}$$

LEMMA 1.2 Let a be a nonzero ideal in $k[X_1,...,X_n]$; then $(LT(a))$ is a monomial ideal, and it equals $(LT(g_1),...,LT(g_n))$ for some $g_1,...,g_n \in a$.

PROOF. Since $(LT(a))$ can also be described as the ideal generated by the leading monomials (rather than the leading terms) of elements of a.

THEOREM 1.2. Every *ideal* a in $k[X_1,...,X_n]$ is finitely generated; more precisely, $a = (g_1,...,g_s)$ where $g_1,...,g_s$ are any elements of a whose leading terms generate $LT(a)$

PROOF. Let $f \in a$. On applying the division algorithm, we find $f = a_1 g_1 + ... + a_s g_s + r, \quad a_i, r \in k[X_1, ..., X_n]$, where either $r = 0$ or no monomial occurring in it is divisible by any $LT(g_i)$. But $r = f - \sum a_i g_i \in a$, and therefore $LT(r) \in LT(a) = (LT(g_1), ..., LT(g_s))$, implies that every monomial occurring in r is divisible by one in $LT(g_i)$. Thus $r = 0$, and $g \in (g_1, ..., g_s)$.

DEFINITION 1.1. A finite subset $S = \{g_1, |..., g_s\}$ of an ideal a is a standard ($(Gr\ddot{o}bner)$ bases for a if $(LT(g_1), ..., LT(g_s)) = LT(a)$. In other words, S is a standard basis if the leading term of every element of a is divisible by at least one of the leading terms of the g_i.

THEOREM 1.3 *The ring* $k[X_1, ..., X_n]$ *is Noetherian i.e., every ideal is finitely generated.*

PROOF. For $n = 1$, $k[X]$ is a principal ideal domain, which means that every ideal is generated by single element. We shall prove the theorem by induction on n. Note that the obvious map $k[X_1, ... X_{n-1}][X_n] \to k[X_1, ... X_n]$ is an isomorphism – this simply says that every polynomial f in n variables $X_1, ... X_n$ can be expressed uniquely as a polynomial in X_n with coefficients in $k[X_1, ..., X_n]$:

$$f(X_1, ... X_n) = a_0(X_1, ... X_{n-1}) X_n^r + ... + a_r(X_1, ... X_{n-1})$$

Thus the next lemma will complete the proof

LEMMA 1.3. If A is Noetherian, then so also is $A[X]$

PROOF. For a polynomial

$$f(X) = a_0 X^r + a_1 X^{r-1} + ... + a_r, \quad a_i \in A, \quad a_0 \neq 0,$$

r is called the degree of f, and a_0 is its leading coefficient. We call 0 the leading coefficient of the polynomial 0. Let a be an ideal in $A[X]$. The leading coefficients of the polynomials in a form an ideal a' in A, and since A is Noetherian, a' will be finitely generated. Let $g_1, ..., g_m$ be elements of a whose leading coefficients generate a', and let r be the maximum degree of g_i. Now let $f \in a$, and suppose f has degree $s > r$, say, $f = aX^s + ...$ Then $a \in a'$, and so we can write

$$a = \sum b_i a_i, \qquad b_i \in A,$$

$a_i = \textit{leading coefficient of } g_i$

Now

$f - \sum b_i g_i X^{s - r_i}$, $r_i = \deg(g_i)$, has degree $< \deg(f)$. By continuing in this way, we find that $f \equiv f_t \mod(g_1, ... g_m)$ With f_t a polynomial of degree $t < r$. For each $d < r$, let a_d be the subset of A consisting of 0 and the leading coefficients of all polynomials in a of degree d; it is again an ideal in A. Let $g_{d,1}, ..., g_{d,m_d}$ be polynomials of degree d whose leading coefficients generate a_d. Then the same argument as above shows that any polynomial f_d in a of degree d can be written $f_d \equiv f_{d-1} \mod(g_{d,1}, ... g_{d,m_d})$ With f_{d-1} of degree $\leq d-1$. On applying this remark repeatedly we find that $f_t \in (g_{r-1,1}, ... g_{r-1,m_{r-1}}, ... g_{0,1}, ... g_{0,m_0})$ Hence

$$f_t \in (g_1, ... g_m g_{r-1,1}, ... g_{r-1,m_{r-1}}, ..., g_{0,1}, ..., g_{0,m_0})$$

and so the polynomials $g_1, ..., g_{0,m_0}$ generate a

One of the great successes of category theory in computer science has been the development of a "unified theory" of the constructions underlying denotational semantics. In the untyped λ-calculus, any term may appear in the function position of an application. This means that a model D of the λ-calculus must have the property that given a term t whose interpretation is $d \in D$, Also, the interpretation of a functional abstraction like $\lambda x \,.\, x$ is most conveniently defined as a function from $D to D$, which must then be regarded as an element of D. Let $\psi : [D \to D] \to D$ be the function that picks out elements of D to represent elements of $[D \to D]$

and $\phi : D \to [D \to D]$ be the function that maps elements of *D* to functions of *D*. Since $\psi(f)$ is intended to represent the function f as an element of *D*, it makes sense to require that $\phi(\psi(f)) = f$, that is, $\psi\, o\, \psi = id_{[D \to D]}$ Furthermore, we often want to view every element of *D* as representing some function from *D to D* and require that elements representing the same function be equal – that is

$$\psi(\varphi(d)) = d$$

or

$$\psi\, o\, \phi = id_D$$

The latter condition is called extensionality. These conditions together imply that $\phi\, and\, \psi$ are inverses--- that is, *D* is isomorphic to the space of functions from *D to D* that can be the interpretations of functional abstractions: $D \cong [D \to D]$ .Let us suppose we are working with the untyped $\lambda - calculus$, we need a solution ot the equation $D \cong A + [D \to D]$, where A is some predetermined domain containing interpretations for elements of *C*. Each element of *D* corresponds to either an element of *A* or an element of $[D \to D]$, with a tag. This equation can be solved by finding least fixed points of the function $F(X) = A + [X \to X]$ from domains to domains --- that is, finding domains X such that $X \cong A + [X \to X]$, and such that for any domain *Y* also satisfying this equation, there is an embedding of *X* to *Y* --- a pair of maps

$$X \underset{f^R}{\overset{f}{\rightleftarrows}} Y$$

Such that

$$f^R\, o\, f = id_X$$

$$f\, o\, f^R \subseteq id_Y$$

Where $f \subseteq g$ means that $f\, approximates\, g$ in some ordering representing their information content. The key shift of perspective from the domain-theoretic to the more general category-theoretic approach lies in considering *F* not as a function on domains, but as a *functor* on a category of domains. Instead of a least fixed point of the function, *F.*

Definition 1.3: Let ***K*** be a category and $F : K \to K$ as a functor. A fixed point of *F* is a pair (A,a), where A is a ***K-object*** and $a : F(A) \to A$ is an isomorphism. A prefixed point of F is a pair (A,a), where A is a ***K-object*** and a is any arrow from F(A) to A

Definition 1.4 : An $\omega - chain$ in a category ***K*** is a diagram of the following form:

$$\Delta = D_o \xrightarrow{f_o} D_1 \xrightarrow{f_1} D_2 \xrightarrow{f_2} \ldots\ldots$$

Recall that a cocone μ of an $\omega - chain$ Δ is a *K*-object X and a collection of K –*arrows* $\{\mu_i : D_i \to X \mid i \geq 0\}$ such that $\mu_i = \mu_{i+1} o\, f_i$ for all $i \geq 0$. We sometimes write $\mu : \Delta \to X$ as a reminder of the arrangement of $\mu' s$ components Similarly, a colimit $\mu : \Delta \to X$ is a cocone with the property that if $\nu : \Delta \to X'$ is also a cocone then there exists a unique mediating arrow $k : X \to X'$ such that for all $i \geq 0,$, $\nu_i = k\, o\, \mu_i$. Colimits of $\omega - chains$ are sometimes referred to as $\omega - co\lim its$. Dually, an $\omega^{op} - chain$ in ***K*** is a diagram of the following form:

$$\Delta = D_o \xleftarrow{f_o} D_1 \xleftarrow{f_1} D_2 \xleftarrow{f_2} \ldots\ldots$$

A cone $\mu : X \to \Delta$ of an $\omega^{op} - chain$ Δ is a **K**-object X and a collection of **K**-arrows $\{\mu_i : D_i \mid i \geq 0\}$ such that for all $i \geq 0$, $\mu_i = f_i\, o\, \mu_{i+1}$. An ω^{op} -limit of an $\omega^{op} - chain$ Δ is a cone $\mu : X \to \Delta$ with the property that if $\nu : X' \to \Delta$ is also a cone, then there exists a unique mediating arrow $k : X' \to X$ such that for all $i \geq 0, \mu_i\, o\, k = \nu_i$. We write $\perp_k$ (or just $\perp$) for the distinguish initial object of ***K,*** when it has one, and $\perp \to A$ for the unique arrow from $\perp$ to each ***K***-object A. It is also convenient to write $\Delta^- = D_1 \xrightarrow{f_1} D_2 \xrightarrow{f_2} \ldots\ldots$ to denote all of Δ except D_o and f_0. By analogy, μ^- is $\{\mu_i \mid i \geq 1\}$. For the images of Δ and μ under ***F*** we write

$$F(\Delta) = F(D_o) \xrightarrow{F(f_o)} F(D_1) \xrightarrow{F(f_1)} F(D_2) \xrightarrow{F(f_2)} \ldots\ldots$$

and $F(\mu) = \{F(\mu_i) \mid i \geq 0\}$

We write F^i for the ***i***-fold iterated composition of ***F*** – that is,

$$F^o(f) = f, F^1(f) = F(f), F^2(f) = F(F(f))$$

,etc. With these definitions we can state that every monitonic function on a complete lattice has a least fixed point:

Lemma 1.4. Let K be a category with initial object $\perp$ and let $F: K \rightarrow K$ be a functor. Define the $\omega-chain\,\Delta$ by

$$\Delta = \perp \xrightarrow{!\perp \rightarrow F(\perp)} F(\perp) \xrightarrow{F(!\perp \rightarrow F(\perp))} F^2(\perp) \xrightarrow{F^2(!\perp \rightarrow F(\perp))} \ldots\ldots\ldots$$

If both $\mu : \Delta \rightarrow D$ and $F(\mu): F(\Delta) \rightarrow F(D)$ are colimits, then (D,d) is an intial F-algebra, where $d: F(D) \rightarrow D$ is the mediating arrow from $F(\mu)$ to the cocone μ^{-}

Theorem 1.4 Let a DAG G given in which each node is a random variable, and let a discrete conditional probability distribution of each node given values of its parents in G be specified. Then the product of these conditional distributions yields a joint probability distribution P of the variables, and (G,P) satisfies the Markov condition.

Proof. Order the nodes according to an ancestral ordering. Let $X_1, X_2, \ldots\ldots.. X_n$ be the resultant ordering. Next define.

$$P(x_1, x_2, \ldots. x_n) = P(x_n \mid pa_n) P(x_{n-1} \mid Pa_{n-1}) \ldots$$
$$..P(x_2 \mid pa_2) P(x_1 \mid pa_1),$$

Where PA_i is the set of parents of X_i of in G and $P(x_i \mid pa_i)$ is the specified conditional probability distribution. First we show this does indeed yield a joint probability distribution. Clearly, $0 \leq P(x_1, x_2, \ldots x_n) \leq 1$ for all values of the variables. Therefore, to show we have a joint distribution, as the variables range through all their possible values, is equal to one. To that end, Specified conditional distributions are the conditional distributions they notationally represent in the joint distribution. Finally, we show the Markov condition is satisfied. To do this, we need show for $1 \leq k \leq n$ that
whenever

$$P(pa_k) \neq 0, if \;\; P(nd_k \mid pa_k) \neq 0$$
$$and \;\; P(x_k \mid pa_k) \neq 0$$
$$then \; P(x_k \mid nd_k, pa_k) = P(x_k \mid pa_k),$$

Where ND_k is the set of nondescendents of X_k of in G. Since $PA_k \subseteq ND_k$, we need only show $P(x_k \mid nd_k) = P(x_k \mid pa_k)$. First for a given k, order the nodes so that all and only nondescendents of X_k precede X_k in the ordering. Note that this ordering depends on k, whereas the ordering in the first part of the proof does not. Clearly then

$$ND_k = \{X_1, X_2, \ldots. X_{k-1}\}$$
$$Let$$
$$D_k = \{X_{k+1}, X_{k+2}, \ldots. X_n\}$$

follows $\sum_{d_k}$

We define the m^{th} *cyclotomic field to be the field* $Q[x]/(\Phi_m(x))$ Where $\Phi_m(x)$ is the m^{th} cyclotomic polynomial. $Q[x]/(\Phi_m(x))$ $\Phi_m(x)$ *has degree* $\varphi(m)$ *over* Q *since* $\Phi_m(x)$ has degree $\varphi(m)$. *The roots of* $\Phi_m(x)$ are just the primitive m^{th} roots of unity, so the complex embeddings of $Q[x]/(\Phi_m(x))$ *are simply the* $\varphi(m)$ *maps*

$$\sigma_k : Q[x]/(\Phi_m(x)) \mapsto C,$$
$$1 \leq k \prec m, (k,m) = 1, \quad where$$
$$\sigma_k(x) = \xi_m^k,$$

ξ_m being our fixed choice of primitive m^{th} root of unity. Note that $\xi_m^k \in Q(\xi_m)$ for every k; it follows that $Q(\xi_m) = Q(\xi_m^k)$ for all k relatively prime to m. In particular, the images of the σ_i coincide, so $Q[x]/(\Phi_m(x))$ *is Galois over* Q. *This means that we can write* $Q(\xi_m)$ *for* $Q[x]/(\Phi_m(x))$ *without much fear of ambiguity; we will do so from now on, the identification being* $\xi_m \mapsto x$. *One advantage of this is that one can easily talk about cyclotomic fields being extensions of one another,or intersections or compositums; all of these things take place considering them as subfield of* C. We now investigate some basic properties of cyclotomic fields. The first issue is whether or not they are all distinct; to determine this, we need to know which roots of unity lie in $Q(\xi_m)$ .Note, for example, that if m is odd, then $-\xi_m$ is a $2m^{th}$ root of unity. We will show that this is the only way in which one can obtain any non-m^{th} roots of unity.

LEMMA 1.5 If m divides n, then $Q(\xi_m)$ *is contained in* $Q(\xi_n)$

PROOF. *Since* $\xi^{n/m} = \xi_m$, *we have* $\xi_m \in Q(\xi_n)$, *so the result is clear*

LEMMA 1.6 *If* m and n are relatively prime, then

$$Q(\xi_m,\xi_n)=Q(\xi_{nm})$$

and

$$Q(\xi_m)\cap Q(\xi_n)=Q$$

(Recall the $Q(\xi_m,\xi_n)$ is the compositum of $Q(\xi_m)$ *and* $Q(\xi_n)$)

PROOF. One checks easily that $\xi_m\xi_n$ is a primitive mn^{th} root of unity, so that

$Q(\xi_{mn})\subseteq Q(\xi_m,\xi_n)$

$[Q(\xi_m,\xi_n):Q]\le[Q(\xi_m):Q][Q(\xi_n:Q]$

$=\varphi(m)\varphi(n)=\varphi(mn);$

Since $[Q(\xi_{mn}):Q]=\varphi(mn);$ this implies that $Q(\xi_m,\xi_n)=Q(\xi_{nm})$ We know that $Q(\xi_m,\xi_n)$ has degree $\varphi(mn)$ over Q, so we must have

$$[Q(\xi_m,\xi_n):Q(\xi_m)]=\varphi(n)$$

and

$$[Q(\xi_m,\xi_n):Q(\xi_m)]=\varphi(m)$$

$$[Q(\xi_m):Q(\xi_m)\cap Q(\xi_n)]\ge\varphi(m)$$

And thus that $Q(\xi_m)\cap Q(\xi_n)=Q$

PROPOSITION 1.2 For any m and n

$Q(\xi_m,\xi_n)=Q(\xi_{[m,n]})$

And

$Q(\xi_m)\cap Q(\xi_n)=Q(\xi_{(m,n)});$

here $[m,n]$ and (m,n) denote the least common multiple and the greatest common divisor of m and n, respectively.

PROOF. Write $m=p_1^{e_1}......p_k^{e_k}$ *and* $p_1^{f_1}....p_k^{f_k}$ where the p_i are distinct primes. (We allow e_i *or* f_i to be zero)

$Q(\xi_m)=Q(\xi_{p_1^{e_1}})Q(\xi_{p_2^{e_2}})...Q(\xi_{p_k^{e_k}})$

and

$Q(\xi_n)=Q(\xi_{p_1^{f_1}})Q(\xi_{p_2^{f_2}})...Q(\xi_{p_k^{f_k}})$

Thus

$$\begin{aligned}Q(\xi_m,\xi_n)&=Q(\xi_{p_1^{e_1}})........Q(\xi_{p_2^{e_k}})Q(\xi_{p_1^{f_1}})...Q(\xi_{p_k^{f_k}})\\&=Q(\xi_{p_1^{e_1}})Q(\xi_{p_1^{f_1}})...Q(\xi_{p_k^{e_k}})Q(\xi_{p_k^{f_k}})\\&=Q(\xi_{p_1^{\max(e_1,f_1)}}).......Q(\xi_{p_1^{\max(e_k,f_k)}})\\&=Q(\xi_{p_1^{\max(e_1,f_1)}........p_1^{\max(e_k,f_k)}})\\&=Q(\xi_{[m,n]});\end{aligned}$$

An entirely similar computation shows that $Q(\xi_m)\cap Q(\xi_n)=Q(\xi_{(m,n)})$

Mutual information measures the information transferred when x_i is sent and y_i is received, and is defined as

$$I(x_i,y_i)=\log_2\frac{P(x_i/y_i)}{P(x_i)}\ bits \qquad (1)$$

In a noise-free channel, **each** y_i is uniquely connected to the corresponding x_i , and so they constitute an input –output pair (x_i,y_i) for which $P(x_i/y_j)=1$ *and* $I(x_i,y_j)=\log_2\frac{1}{P(x_i)}$ bits; that is, the transferred information is equal to the self-information that corresponds to the input x_i In a very noisy channel, the output y_i and input x_i would be completely uncorrelated, and so $P(x_i/y_j)=P(x_i)$ and also $I(x_i,y_j)=0;$ that is, there is no transference of information. In general, a given channel will operate between these two extremes. The mutual information is defined between the input and the output of a given channel. An average of the calculation of the mutual information for all input-output pairs of a given channel is the average mutual information:

$$I(X,Y)=\sum_{i.j}P(x_i,y_j)I(x_i,y_j)=\sum_{i.j}P(x_i,y_j)\log_2\left[\frac{P(x_i/y_j)}{P(x_i)}\right]$$

bits per symbol . This calculation is done over the input and output alphabets. The average mutual information. The following expressions are useful for modifying the mutual information expression:

$$P(x_i, y_j) = P(x_i/y_j)P(y_j) = P(y_j/x_i)P(x_i)$$

$$P(y_j) = \sum_i P(y_j/x_i)P(x_i)$$

$$P(x_i) = \sum_i P(x_i/y_j)P(y_j)$$

Then

$$I(X,Y) = \sum_{i.j} P(x_i, y_j)$$

$$= \sum_{i.j} P(x_i, y_j)\log_2\left[\frac{1}{P(x_i)}\right]$$

$$-\sum_{i.j} P(x_i, y_j)\log_2\left[\frac{1}{P(x_i/y_j)}\right]$$

$$\sum_{i.j} P(x_i, y_j)\log_2\left[\frac{1}{P(x_i)}\right]$$

$$= \sum_i\left[P(x_i/y_j)P(y_j)\right]\log_2\frac{1}{P(x_i)}$$

$$\sum_i P(x_i)\log_2\frac{1}{P(x_i)} = H(X)$$

$$I(X,Y) = H(X) - H(X/Y)$$

Where

$$H(X/Y) = \sum\nolimits_{i,j} P(x_i, y_j)\log_2\frac{1}{P(x_i/y_j)}$$ is

usually called the equivocation. In a sense, the equivocation can be seen as the information lost in the noisy channel, and is a function of the backward conditional probability. The observation of an output symbol y_j provides $H(X) - H(X/Y)$ bits of information. This difference is the mutual information of the channel. *Mutual Information: Properties* Since

$$P(x_i/y_j)P(y_j) = P(y_j/x_i)P(x_i)$$

The mutual information fits the condition

$$I(X,Y) = I(Y,X)$$

And by interchanging input and output it is also true that

$$I(X,Y) = H(Y) - H(Y/X)$$

Where

$$H(Y) = \sum_j P(y_j)\log_2\frac{1}{P(y_j)}$$

This last entropy is usually called the noise entropy. Thus, the information transferred through the channel is the difference between the output entropy and the noise entropy. Alternatively, it can be said that the channel mutual information is the difference between the number of bits needed for determining a given input symbol before knowing the corresponding output symbol, and the number of bits needed for determining a given input symbol after knowing the corresponding output symbol

$$I(X,Y) = H(X) - H(X/Y)$$

As the channel mutual information expression is a difference between two quantities, it seems that this parameter can adopt negative values. However, and is spite of the fact that for some $y_j, H(X / y_j)$ can be larger than $H(X)$, this is not possible for the average value calculated over all the outputs:

$$\sum_{i,j} P(x_i, y_j)\log_2\frac{P(x_i/y_j)}{P(x_i)} = \sum_{i,j} P(x_i, y_j)\log_2\frac{P(x_i, y_j)}{P(x_i)P(y_j)}$$

Then

$$-I(X,Y) = \sum_{i,j} P(x_i, y_j)\frac{P(x_i)P(y_j)}{P(x_i, y_j)} \le 0$$

Because this expression is of the form

$$\sum_{i=1}^{M} P_i\log_2\left(\frac{Q_i}{P_i}\right) \le 0$$

The above expression can be applied due to the factor $P(x_i)P(y_j)$, which is the product of two probabilities, so that it behaves as the quantity Q_i, which in this expression is a dummy variable that fits the condition $\sum_i Q_i \le 1$. It can be concluded that the average mutual information is a non-negative number. It can also be equal to zero, when the input and the output are independent of each other. A related entropy called the joint entropy is defined as

$$H(X,Y) = \sum_{i,j} P(x_i, y_j)\log_2\frac{1}{P(x_i, y_j)}$$

$$= \sum_{i,j} P(x_i, y_j)\log_2\frac{P(x_i)P(y_j)}{P(x_i, y_j)}$$

$$+\sum_{i,j} P(x_i, y_j)\log_2\frac{1}{P(x_i)P(y_j)}$$

Theorem 1.5: Entropies of the binary erasure channel (BEC) The BEC is defined with an alphabet of two inputs and three outputs, with symbol probabilities.

$P(x_1) = \alpha \ \ and \ \ P(x_2) = 1 - \alpha$, and transition probabilities

$$P({}^{y_3}/_{x_2}) = 1 - p \ \ and \ \ P({}^{y_2}/_{x_1}) = 0,$$
$$and \ \ P({}^{y_3}/_{x_1}) = 0$$
$$and \ \ P({}^{y_1}/_{x_2}) = p$$
$$and \ \ P({}^{y_3}/_{x_2}) = 1 - p$$

Lemma 1.7. Given an arbitrary restricted time-discrete, amplitude-continuous channel whose restrictions are determined by sets F_n and whose density functions exhibit no dependence on the state s, let n be a fixed positive integer, and $p(x)$ an arbitrary probability density function on Euclidean *n*-space. $p(y|x)$ for the density $p_n(y_1,...,y_n | x_1,...x_n)$ and F *for* F_n. For any real number a, let

$$A = \left\{(x,y) : \log \frac{p(y|x)}{p(y)} > a\right\} \qquad (1)$$

Then for each positive integer u, there is a code (u,n,λ) such that

$$\lambda \le ue^{-a} + P\{(X,Y) \notin A\} + P\{X \notin F\} \qquad (2)$$

Where

$$P\{(X,Y) \in A\} = \int_A ... \int p(x,y)dxdy, \qquad p(x,y) = p(x)p(y|x)$$

and

$$P\{X \in F\} = \int_F ... \int p(x)dx$$

Proof: A sequence $x^{(1)} \in F$ *such that*

$$P\{Y \in A_{x^1} | X = x^{(1)}\} \ge 1 - \varepsilon$$

$$where \ A_x = \{y : (x,y) \varepsilon A\};$$

Choose the decoding set B_1 to be $A_{x^{(1)}}$. Having chosen $x^{(1)},........,x^{(k-1)}$ and $B_1,...,B_{k-1}$, select $x^k \in F$ such that

$$P\left\{Y \in A_{x^{(k)}} - \bigcup_{i=1}^{k-1} B_i | X = x^{(k)}\right\} \ge 1 - \varepsilon;$$

Set $B_k = A_{x^{(k)}} - \bigcup_{i=1}^{k-1} B_i$, If the process does not terminate in a finite number of steps, then the sequences $x^{(i)}$ and decoding sets B_i, $i = 1, 2, ..., u$, form the desired code. Thus assume that the process terminates after t steps. (Conceivably $t = 0$). We will show $t \ge u$ by showing that $\varepsilon \le te^{-a} + P\{(X,Y) \notin A\} + P\{X \notin F\}$. We proceed as follows.

Let

$$B = \bigcup_{j=1}^{t} B_j. \quad (If \ \ t = 0, \ take \ \ B = \phi). \ \ Then$$

$$P\{(X,Y) \in A\} = \int_{(x,y)\in A} p(x,y)dx\,dy$$

$$= \int_x p(x) \int_{y \in A_x} p(y|x)dy\,dx$$

$$= \int_x p(x) \int_{y \in B \cap A_x} p(y|x)dy\,dx + \int_x p(x)$$

C. Algorithms

Ideals. Let A be a ring. Recall that an *ideal a* in A is a subset such that a is subgroup of A regarded as a group under addition;

$$a \in a, r \in A \Rightarrow ra \in A$$

The ideal generated by a subset S of A is the intersection of all ideals A containing a ----- it is easy to verify that this is in fact an ideal, and that it consist of all finite sums of the form $\sum r_i s_i$ with $r_i \in A, s_i \in S$. When $S = \{s_1,.....,s_m\}$, we shall write $(s_1,.....,s_m)$ for the ideal it generates.

Let a and b be ideals in A. The set $\{a+b | a \in a, b \in b\}$ is an ideal, denoted by $a+b$. The ideal generated by $\{ab | a \in a, b \in b\}$ is denoted by ab. Note that $ab \subset a \cap b$. Clearly ab consists of all finite sums $\sum a_i b_i$ with $a_i \in a$ and $b_i \in b$, and if $a = (a_1,...,a_m)$ and $b = (b_1,...,b_n)$, then $ab = (a_1 b_1,...,a_i b_j,...,a_m b_n)$ .Let a be an ideal of A. The set of cosets of a in A forms a ring A/a, and $a \mapsto a + a$ is a homomorphism $\phi : A \mapsto A/a$. The map $b \mapsto \phi^{-1}(b)$ is a one to one correspondence between the ideals of A/a and the ideals of A containing a An ideal p if *prime* if $p \ne A$ and $ab \in p \Rightarrow a \in p$ or $b \in p$. Thus p is prime if and only if A/p is nonzero and has the property that $ab = 0, \quad b \ne 0 \Rightarrow a = 0,$ i.e., A/p is an integral domain. An ideal m is *maximal* if $m \ne| A$ and there does not exist an ideal n contained strictly between m and A. Thus m is maximal if and only if A/m has no proper nonzero ideals, and so is a field. Note that m maximal $\Rightarrow$ m prime. The ideals of $A \times B$ are all of the form $a \times b$, with a and b ideals in A and B. To see this, note that if c is an ideal in $A \times B$ and $(a,b) \in c$, then $(a,0) = (a,b)(1,0) \in c$ and $(0,b) = (a,b)(0,1) \in c$. This shows that $c = a \times b$ with

$a = \{a \mid (a,b) \in c \ \ some \ \ b \in b\}$

and

$b = \{b \mid (a,b) \in c \ \ some \ \ a \in a\}$

Let A be a ring. An A-algebra is a ring B together with a homomorphism $i_B : A \to B$. A *homomorphism of* A -algebra $B \to C$ is a homomorphism of rings $\varphi : B \to C$ such that $\varphi(i_B(a)) = i_C(a)$ for all $a \in A$. An A-algebra B is said to be *finitely generated* (or of *finite-type* over A) if there exist elements $x_1,..., x_n \in B$ such that every element of B can be expressed as a polynomial in the x_i with coefficients in $i(A)$, i.e., such that the homomorphism $A[X_1,..., X_n] \to B$ sending X_i to x_i is surjective. A ring homomorphism $A \to B$ is *finite*, and B is finitely generated as an A-module. Let k be a field, and let A be a k-algebra. If $1 \neq 0$ in A, then the map $k \to A$ is injective, we can identify k with its image, i.e., we can regard k as a subring of A. If 1=0 in a ring R, the R is the zero ring, i.e., $R = \{0\}$. **Polynomial rings.** Let k be a field. A *monomial* in $X_1,..., X_n$ is an expression of the form $X_1^{a_1}...X_n^{a_n}$, $a_j \in N$. The *total degree* of the monomial is $\sum a_i$. We sometimes abbreviate it by X^α, $\alpha = (a_1,..., a_n) \in \mathbb{N}^n$. The elements of the polynomial ring $k[X_1,..., X_n]$ are finite sums $\sum c_{a_1....a_n} X_1^{a_1}...X_n^{a_n}$, $c_{a_1....a_n} \in k$, $a_j \in \mathbb{N}$ With the obvious notions of equality, addition and multiplication. Thus the monomials from basis for $k[X_1,..., X_n]$ as a k-vector space. The ring $k[X_1,..., X_n]$ is an integral domain, and the only units in it are the nonzero constant polynomials. A polynomial $f(X_1,..., X_n)$ is *irreducible* if it is nonconstant and has only the obvious factorizations, i.e., $f = gh \Rightarrow g$ or h is constant. **Division in** $k[X]$. The division algorithm allows us to divide a nonzero polynomial into another: let f and g be polynomials in $k[X]$ with $g \neq 0$; then there exist unique polynomials $q, r \in k[X]$ such that $f = qg + r$ with either $r = 0$ or deg r < deg g. Moreover, there is an algorithm for deciding whether $f \in (g)$, namely, find r and check whether it is zero. Moreover, the Euclidean algorithm allows to pass from finite set of generators for an ideal in $k[X]$ to a single generator by successively replacing each pair of generators with their greatest common divisor.

(Pure) ***lexicographic*** *ordering (lex).* Here monomials are ordered by lexicographic(dictionary) order. More precisely, let $\alpha = (a_1,...a_n)$ and $\beta = (b_1,...b_n)$ be two elements of $\mathbb{N}^n$; then $\alpha > \beta$ *and* $X^\alpha > X^\beta$ (lexicographic ordering) if, in the vector difference $\alpha - \beta \in \mathbb{Z}$, the left most nonzero entry is positive. For example,

$XY^2 > Y^3Z^4; \quad X^3Y^2Z^4 > X^3Y^2Z$. Note that this isn't quite how the dictionary would order them: it would put $XXXYYZZZZ$ after $XXXYYZ$. *Graded reverse lexicographic order (grevlex).* Here monomials are ordered by total degree, with ties broken by reverse lexicographic ordering. Thus, $\alpha > \beta$ if $\sum a_i > \sum b_i$, or $\sum a_i = \sum b_i$ and in $\alpha - \beta$ the right most nonzero entry is negative. For example:

$X^4Y^4Z^7 > X^5Y^5Z^4$ *(total degree greater)*

$XY^5Z^2 > X^4YZ^3, \quad X^5YZ > X^4YZ^2$.

Orderings on $k[X_1,...X_n]$. Fix an ordering on the monomials in $k[X_1,...X_n]$. Then we can write an element f of $k[X_1,...X_n]$ in a canonical fashion, by re-ordering its elements in decreasing order. For example, we would write

$f = 4XY^2Z + 4Z^2 - 5X^3 + 7X^2Z^2$

as

$f = -5X^3 + 7X^2Z^2 + 4XY^2Z + 4Z^2 \quad (lex)$

or

$f = 4XY^2Z + 7X^2Z^2 - 5X^3 + 4Z^2 \quad (grevlex)$

Let $\sum a_\alpha X^\alpha \in k[X_1,\ldots,X_n]$, in decreasing order:

$$f = a_{\alpha_0} X^{\alpha_0} + _{\alpha_1} X^{\alpha_1} + \ldots, \qquad \alpha_0 > \alpha_1 > \ldots, \quad \alpha_0 \neq 0$$

Then we define.

- The *multidegree* of f to be multdeg(f)= α_0;
- The *leading coefficient of* f to be *LC*(f)= a_{α_0};
- The *leading monomial of* f to be *LM*(f) = X^{α_0};
- The *leading term of* f to be *LT*(f) = $a_{\alpha_0} X^{\alpha_0}$

For the polynomial $f = 4XY^2Z + \ldots,$ the multidegree is (1,2,1), the leading coefficient is 4, the leading monomial is XY^2Z, and the leading term is $4XY^2Z$. **The division algorithm in** $k[X_1,\ldots X_n]$. Fix a monomial ordering in $\mathbb{N}^2$. Suppose given a polynomial f and an ordered set $(g_1,\ldots g_s)$ of polynomials; the division algorithm then constructs polynomials $a_1,\ldots a_s$ and r such that $f = a_1g_1 + \ldots + a_sg_s + r$ Where either $r = 0$ or no monomial in r is divisible by any of $LT(g_1),\ldots,LT(g_s)$ **Step 1:** If $LT(g_1) \mid LT(f)$, divide g_1 into f to get

$$f = a_1g_1 + h, \qquad a_1 = \frac{LT(f)}{LT(g_1)} \in k[X_1,\ldots,X_n]$$

If $LT(g_1) \mid LT(h)$, repeat the process until $f = a_1g_1 + f_1$ (different a_1) with $LT(f_1)$ not divisible by $LT(g_1)$. Now divide g_2 into f_1, and so on, until $f = a_1g_1 + \ldots + a_sg_s + r_1$ With $LT(r_1)$ not divisible by any $LT(g_1),\ldots LT(g_s)$ **Step 2:** Rewrite $r_1 = LT(r_1) + r_2$, and repeat Step 1 with r_2 for f : $f = a_1g_1 + \ldots + a_sg_s + LT(r_1) + r_3$ (different a_i's) **Monomial ideals.** In general, an ideal a will contain a polynomial without containing the individual terms of the polynomial; for example, the ideal $a = (Y^2 - X^3)$ contains $Y^2 - X^3$ but not Y^2 or X^3.

DEFINITION 1.5. An ideal a is *monomial* if

$$\sum c_\alpha X^\alpha \in a \Rightarrow X^\alpha \in a$$

all α with $c_\alpha \neq 0$.

PROPOSITION 1.3. Let a be a *monomial ideal*, and let $A = \{\alpha \mid X^\alpha \in a\}$. Then A satisfies the condition $\alpha \in A,\ \beta \in \mathbb{N}^n \Rightarrow \alpha + \beta \in$ $(*)$ And a is the k -subspace of $k[X_1,\ldots,X_n]$ generated by the $X^\alpha, \alpha \in A$. Conversely, of A is a subset of $\mathbb{N}^n$ satisfying $(*)$, then the k-subspace a of $k[X_1,\ldots,X_n]$ generated by $\{X^\alpha \mid \alpha \in A\}$ is a monomial ideal.

PROOF. It is clear from its definition that a monomial ideal a is the k -subspace of $k[X_1,\ldots,X_n]$ generated by the set of monomials it contains. If $X^\alpha \in a$ and $X^\beta \in k[X_1,\ldots,X_n]$.

If a permutation is chosen uniformly and at random from the $n!$ possible permutations in S_n, then the counts $C_j^{(n)}$ of cycles of length j are dependent random variables. The joint distribution of $C^{(n)} = (C_1^{(n)},\ldots,C_n^{(n)})$ follows from Cauchy's formula, and is given by

$$P[C^{(n)} = c] = \frac{1}{n!} N(n,c) = 1\left\{\sum_{j=1}^{n} jc_j = n\right\} \prod_{j=1}^{n} \left(\frac{1}{j}\right)^{c_j} \frac{1}{c_j!}, \qquad (1.1)$$

for $c \in \mathbb{N}_+^n$.

Lemma1.7 For nonnegative integers $m_{1,\ldots,}m_n$,

$$E\left(\prod_{j=1}^{n} (C_j^{(n)})^{[m_j]}\right) = \left(\prod_{j=1}^{n} \left(\frac{1}{j}\right)^{m_j}\right) 1\left\{\sum_{j=1}^{n} jm_j \le n\right\} \qquad (1.4)$$

Proof. This can be established directly by exploiting cancellation of the form $c_j^{[m_j]} / c_j^! = 1/(c_j - m_j)!$ when $c_j \ge m_j$, which occurs between the ingredients in Cauchy's formula and the falling factorials in the moments. Write $m = \sum jm_j$. Then, with the first sum indexed by $c = (c_1,\ldots c_n) \in \mathbb{N}_+^n$ and the last sum indexed by $d = (d_1,\ldots,d_n) \in \mathbb{N}_+^n$ via the correspondence $d_j = c_j - m_j$, we have

$$E\left(\prod_{j=1}^{n}(C_j^{(n)})^{[m_j]}\right)=\sum_{c}P[C^{(n)}=c]\prod_{j=1}^{n}(c_j)^{[m_j]}$$

$$=\sum_{c:c_j\ge m_j \text{ for all } j}1\left\{\sum_{j=1}^{n}jc_j=n\right\}\prod_{j=1}^{n}\frac{(c_j)^{[m_j]}}{j^{c_j}c_j!}$$

$$=\prod_{j=1}^{n}\frac{1}{j^{m_j}}\sum_{d}1\left\{\sum_{j=1}^{n}jd_j=n-m\right\}\prod_{j=1}^{n}\frac{1}{j^{d_j}(d_j)!}$$

This last sum simplifies to the indicator $1(m\le n)$, corresponding to the fact that if $n-m\ge 0$, then $d_j=0$ for $j>n-m$, and a random permutation in S_{n-m} must have some cycle structure $(d_1,\ldots,d_{n-m})$. The moments of $C_j^{(n)}$ follow immediately as

$$E(C_j^{(n)})^{[r]}=j^{-r}1\{jr\le n\} \qquad (1.2)$$

We note for future reference that (1.4) can also be written in the form

$$E\left(\prod_{j=1}^{n}(C_j^{(n)})^{[m_j]}\right)=E\left(\prod_{j=1}^{n}Z_j^{[m_j]}\right)1\left\{\sum_{j=1}^{n}jm_j\le n\right\}, \qquad (1.3)$$

Where the Z_j are independent Poisson-distribution random variables that satisfy $E(Z_j)=1/j$

The marginal distribution of cycle counts provides a formula for the joint distribution of the cycle counts C_j^n, we find the distribution of C_j^n using a combinatorial approach combined with the inclusion-exclusion formula.

Lemma 1.8. For $1\le j\le n$,

$$P[C_j^{(n)}=k]=\frac{j^{-k}}{k!}\sum_{l=0}^{[n/j]-k}(-1)^l\frac{j^{-l}}{l!} \qquad (1.1)$$

Proof. Consider the set I of all possible cycles of length j, formed with elements chosen from $\{1,2,\ldots n\}$, so that $|I|=n^{[j]/j}$. For each $\alpha\in I$, consider the "property" G_α of having α; that is, G_α is the set of permutations $\pi\in S_n$ such that α is one of the cycles of π. We then have $|G_\alpha|=(n-j)!$, since the elements of $\{1,2,\ldots,n\}$ not in α must be permuted among themselves. To use the inclusion-exclusion formula we need to calculate the term S_r, which is the sum of the probabilities of the r-fold intersection of properties, summing over all sets of r distinct properties. There are two cases to consider. If the r properties are indexed by r cycles having no elements in common, then the intersection specifies how rj elements are moved by the permutation, and there are $(n-rj)!1(rj\le n)$ permutations in the intersection. There are $n^{[rj]}/(j^r r!)$ such intersections. For the other case, some two distinct properties name some element in common, so no permutation can have both these properties, and the r-fold intersection is empty. Thus

$$S_r=(n-rj)!1(rj\le n)$$

$$\times\frac{n^{[rj]}}{j^r r!}\frac{1}{n!}=1(rj\le n)\frac{1}{j^r r!}$$

Finally, the inclusion-exclusion series for the number of permutations having exactly k properties is

$$\sum_{l\ge 0}(-1)^l\binom{k+l}{l}S_{k+l},$$

Which simplifies to (1.1) Returning to the original hat-check problem, we substitute j=1 in (1.1) to obtain the distribution of the number of fixed points of a random permutation. For $k=0,1,\ldots,n,$

$$P[C_1^{(n)}=k]=\frac{1}{k!}\sum_{l=0}^{n-k}(-1)^l\frac{1}{l!}, \qquad (1.2)$$

and the moments of $C_1^{(n)}$ follow from (1.2) with $j=1$. In particular, for $n\ge 2$, the mean and variance of $C_1^{(n)}$ are both equal to 1. The joint distribution of $(C_1^{(n)},\ldots,C_b^{(n)})$ for any $1\le b\le n$ has an expression similar to (1.7); this too can be derived by inclusion-exclusion. For any $c=(c_1,\ldots,c_b)\in\mathbb{Z}_+^b$ with $m=\sum ic_i$,

$$P[(C_1^{(n)},\ldots,C_b^{(n)})=c]$$

$$=\left\{\prod_{i=1}^{b}\left(\frac{1}{i}\right)^{c_i}\frac{1}{c_i!}\right\}\sum_{\substack{l\ge 0 \text{ with}\\ \sum il_i\le n-m}}(-1)^{l_1+\ldots+l_b}\prod_{i=1}^{b}\left(\frac{1}{i}\right)^{l_i}\frac{1}{l_i!} \qquad (1.3)$$

The joint moments of the first b counts $C_1^{(n)},\ldots,C_b^{(n)}$ can be obtained directly from (1.2) and (1.3) by setting $m_{b+1}=\ldots=m_n=0$

The limit distribution of cycle counts

It follows immediately from Lemma 1.2 that for each fixed j, as $n\to\infty$,

$$P[C_j^{(n)}=k]\to\frac{j^{-k}}{k!}e^{-1/j}, \quad k=0,1,2,\ldots,$$

So that $C_j^{(n)}$ converges in distribution to a random variable Z_j having a Poisson distribution with mean $1/j$; we use the notation $C_j^{(n)}\to_d Z_j$

where $Z_j \sim P_o(1/j)$ to describe this. Infact, the limit random variables are independent.

Theorem 1.6 The process of cycle counts converges in distribution to a Poisson process of $\mathbb{N}$ with intensity j^{-1}. That is, as $n \to \infty$,

$$(C_1^{(n)}, C_2^{(n)}, ...) \to_d (Z_1, Z_2, ...) \quad (1.1)$$

Where the $Z_j, j = 1, 2, ...,$ are independent Poisson-distributed random variables with $E(Z_j) = \frac{1}{j}$

Proof. To establish the converges in distribution one shows that for each fixed $b \geq 1$, as $n \to \infty$,

$$P[(C_1^{(n)}, ..., C_b^{(n)}) = c] \to P[(Z_1, ..., Z_b) = c]$$

Error rates

The proof of Theorem says nothing about the rate of convergence. Elementary analysis can be used to estimate this rate when $b = 1$. Using properties of alternating series with decreasing terms, for $k = 0, 1, ..., n,$

$$\frac{1}{k!}\left(\frac{1}{(n-k+1)!} - \frac{1}{(n-k+2)!}\right) \leq \left|P[C_1^{(n)} = k] - P[Z_1 = k]\right|$$

$$\leq \frac{1}{k!(n-k+1)!}$$

It follows that

$$\frac{2^{n+1}}{(n+1)!}\frac{n}{n+2} \leq \sum_{k=0}^{n} \left|P[C_1^{(n)} = k] - P[Z_1 = k]\right| \leq \frac{2^{n+1}-1}{(n+1)!} \quad (1.11)$$

Since

$$P[Z_1 > n] = \frac{e^{-1}}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + ...\right) < \frac{1}{(n+1)!},$$

We see from (1.11) that the total variation distance between the distribution $L(C_1^{(n)})$ of $C_1^{(n)}$ and the distribution $L(Z_1)$ of Z_1

Establish the asymptotics of $\mathrm{P}\left[A_n(C^{(n)})\right]$ under conditions (A_0) and (B_{01}), where

$$A_n(C^{(n)}) = \bigcap_{1 \leq i \leq n} \bigcap_{r_i^{'}+1 \leq j \leq r_i} \left\{C_{ij}^{(n)} = 0\right\},$$

and $\zeta_i = (r_i^{'}/r_{id}) - 1 = O(i^{-g'})$ as $i \to \infty$, for some $g' > 0$. We start with the expression

$$P[A_n(C^{(n)})] = \frac{P[T_{0m}(Z') = n]}{P[T_{0m}(Z) = n]}$$

$$\prod_{\substack{1 \leq i \leq n \\ r_i^{'}+1 \leq j \leq r_i}} \left\{1 - \frac{\theta}{ir_i}(1 + E_{i0})\right\} \quad (1.1)$$

$$P[T_{0n}(Z') = n]$$

$$= \frac{\theta d}{n}\exp\left\{\sum_{i \geq 1}[\log(1 + i^{-1}\theta d) - i^{-1}\theta d]\right\}$$

$$\left\{1 + O(n^{-1}\varphi_{\{1,2,7\}}^{'}(n))\right\} \quad (1.2)$$

and

$$P[T_{0n}(Z') = n]$$

$$= \frac{\theta d}{n}\exp\left\{\sum_{i \geq 1}[\log(1 + i^{-1}\theta d) - i^{-1}\theta d]\right\}$$

$$\left\{1 + O(n^{-1}\varphi_{\{1,2,7\}}(n))\right\} \quad (1.3)$$

Where $\varphi_{\{1,2,7\}}^{'}(n)$ refers to the quantity derived from Z'. It thus follows that $P[A_n(C^{(n)})] \sim Kn^{-\theta(1-d)}$ for a constant K, depending on Z and the $r_i^{'}$ and computable explicitly from (1.1) – (1.3), if Conditions (A_0) and (B_{01}) are satisfied and if $\zeta_i^* = O(i^{-g'})$ from some $g' > 0$, since, under these circumstances, both $n^{-1}\varphi_{\{1,2,7\}}^{'}(n)$ and $n^{-1}\varphi_{\{1,2,7\}}(n)$ tend to zero as $n \to \infty$. In particular, for polynomials and square free polynomials, the relative error in this asymptotic approximation is of order n^{-1} if $g' > 1$.

For $0 \leq b \leq n/8$ and $n \geq n_0$, with n_0

$$d_{TV}(L(C[1,b]), L(Z[1,b]))$$

$$\leq d_{TV}(L(\tilde{C}[1,b]), L(\tilde{Z}[1,b]))$$

$$\leq \varepsilon_{\{7,7\}}(n,b),$$

Where $\varepsilon_{\{7,7\}}(n,b) = O(b/n)$ under Conditions $(A_0), (D_1)$ and (B_{11}) Since, by the Conditioning Relation,

$$L(\tilde{C}[1,b] \mid T_{0b}(C) = l) = L(\tilde{Z}[1,b] \mid T_{0b}(Z) = l),$$

It follows by direct calculation that

$$d_{TV}(L(\tilde{C}[1,b]), L(\tilde{Z}[1,b]))$$
$$= d_{TV}(L(T_{0b}(C)), L(T_{0b}(Z)))$$
$$= \max_A \sum_{r \in A} P[T_{0b}(Z) = r]$$
$$\left\{1 - \frac{P[T_{bn}(Z) = n - r]}{P[T_{0n}(Z) = n]}\right\} \qquad (1.4)$$

Suppressing the argument Z from now on, we thus obtain

$$d_{TV}(L(\tilde{C}[1,b]), L(\tilde{Z}[1,b]))$$
$$= \sum_{r \ge 0} P[T_{0b} = r]\left\{1 - \frac{P[T_{bn} = n - r]}{P[T_{0n} = n]}\right\}_+$$
$$\le \sum_{r > n/2} P[T_{0b} = r] + \sum_{r=0}^{[n/2]} \frac{P[T_{0b} = r]}{P[T_{0b} = n]}$$
$$\times \left\{\sum_{s=0}^{n} P[T_{0b} = s](P[T_{bn} = n - s] - P[T_{bn} = n - r]\right\}_+$$

$$\le \sum_{r > n/2} P[T_{0b} = r] + \sum_{r=0}^{[n/2]} P[T_{0b} = r]$$
$$\times \sum_{s=0}^{[n/2]} P[T_{0b} = s] \frac{\{P[T_{bn} = n - s] - P[T_{bn} = n - r]\}}{P[T_{0n} = n]}$$
$$+ \sum_{s=0}^{[n/2]} P[T_{0b} = r] \sum_{s=[n/2]+1}^{n} P[T = s]P[T_{bn} = n - s] / P[T_{0n} = n]$$

The first sum is at most $2n^{-1}ET_{0b}$; the third is bound by

$$(\max_{n/2 < s \le n} P[T_{0b} = s]) / P[T_{0n} = n]$$
$$\le \frac{2\varepsilon_{\{10.5(1)\}}(n/2, b)}{n} \frac{3n}{\theta P_\theta[0,1]},$$
$$\frac{3n}{\theta P_\theta[0,1]} 4n^{-2}\phi^*_{\{10.8\}}(n) \sum_{r=0}^{[n/2]} P[T_{0b} = r] \sum_{s=0}^{[n/2]} P[T_{0b} = s] \frac{1}{2}|r - s|$$
$$\le \frac{12\phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]} \frac{ET_{0b}}{n}$$

Hence we may take

$$\varepsilon_{\{7,7\}}(n,b) = 2n^{-1}ET_{0b}(Z)\left\{1 + \frac{6\phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]}\right\}P$$
$$+ \frac{6}{\theta P_\theta[0,1]} \varepsilon_{\{10.5(1)\}}(n/2, b) \qquad (1.5)$$

Required order under Conditions $(A_0), (D_1)$ and (B_{11}), if $S(\infty) < \infty$. If not, $\phi^*_{\{10.8\}}(n)$ can be replaced by $\phi^*_{\{10.11\}}(n)$ in the above, which has the required order, without the restriction on the r_i implied by $S(\infty) < \infty$. Examining the Conditions $(A_0), (D_1)$ and (B_{11}), it is perhaps surprising to find that (B_{11}) is required instead of just (B_{01}); that is, that we should need $\sum_{l \ge 2} l\varepsilon_{il} = O(i^{-a_1})$ to hold for some $a_1 > 1$. A first observation is that a similar problem arises with the rate of decay of ε_{i1} as well. For this reason, n_1 is replaced by $\tilde{n}_1$. This makes it possible to replace condition (A_1) by the weaker pair of conditions (A_0) and (D_1) in the eventual assumptions needed for $\varepsilon_{\{7,7\}}(n,b)$ to be of order $O(b/n)$; the decay rate requirement of order $i^{-1-\gamma}$ is shifted from ε_{i1} itself to its first difference. This is needed to obtain the right approximation error for the random mappings example. However, since all the classical applications make far more stringent assumptions about the $\varepsilon_{i1}, l \ge 2$, than are made in (B_{11}). The critical point of the proof is seen where the initial estimate of the difference $P[T_{bn}^{(m)} = s] - P[T_{bn}^{(m)} = s+1]$. The factor $\varepsilon_{\{10.10\}}(n)$, which should be small, contains a far tail element from $\tilde{n}_1$ of the form $\phi_1^\theta(n) + u_1^*(n)$, which is only small if $a_1 > 1$, being otherwise of order $O(n^{1-a_1+\delta})$ for any $\delta > 0$, since $a_2 > 1$ is in any case assumed. For $s \ge n/2$, this gives rise to a contribution of order $O(n^{-1-a_1+\delta})$ in the estimate of the difference $P[T_{bn} = s] - P[T_{bn} = s+1]$, which, in the remainder of the proof, is translated into a contribution of order $O(tn^{-1-a_1+\delta})$ for differences of the form $P[T_{bn} = s] - P[T_{bn} = s+1]$, finally leading to a contribution of order $bn^{-a_1+\delta}$ for any $\delta > 0$ in $\varepsilon_{\{7.7\}}(n,b)$. Some improvement would seem to be possible, defining the function g by $g(w) = 1_{\{w=s\}} - 1_{\{w=s+t\}}$, differences that are of the form $P[T_{bn} = s] - P[T_{bn} = s+t]$ can be directly estimated, at a cost of only a single contribution of the form $\phi_1^\theta(n) + u_1^*(n)$. Then, iterating the cycle, in which one estimate of a difference in point probabilities is improved to an estimate of smaller order, a bound of the form

$$\left|P[T_{bn}=s]-P[T_{bn}=s+t]\right|=O(n^{-2}t+n^{-1-a_1+\delta})$$

for any $\delta>0$ could perhaps be attained, leading to a final error estimate in order $O(bn^{-1}+n^{-a_1+\delta})$ for any $\delta>0$, to replace $\varepsilon_{\{7.7\}}(n,b)$. This would be of the ideal order $O(b/n)$ for large enough b, but would still be coarser for small b.

With b and n as in the previous section, we wish to show that

$$\left|d_{TV}(L(C[1,b]),L(Z[1,b]))-\frac{1}{2}(n+1)^{-1}\left|1-\theta\right|E\left|T_{0b}-ET_{0b}\right|\right|$$
$$\le\varepsilon_{\{7,8\}}(n,b),$$

Where $\varepsilon_{\{7.8\}}(n,b)=O(n^{-1}b[n^{-1}b+n^{-\beta_{12}+\delta}])$ for any $\delta>0$ under Conditions $(A_0),(D_1)$ and (B_{12}), with β_{12}. The proof uses sharper estimates. As before, we begin with the formula

$$d_{TV}(L(\breve{C}[1,b]),L(\breve{Z}[1,b]))$$
$$=\sum_{r\ge0}P[T_{0b}=r]\left\{1-\frac{P[T_{bn}=n-r]}{P[T_{0n}=n]}\right\}_+$$

Now we observe that

$$\left|\sum_{r\ge0}P[T_{0b}=r]\left\{1-\frac{P[T_{bn}=n-r]}{P[T_{0n}=n]}\right\}_+-\sum_{r=0}^{[n/2]}\frac{P[T_{0b}=r]}{P[T_{0n}=n]}\right|$$
$$\times\left|\sum_{s=[n/2]+1}^{n}P[T_{0b}=s](P[T_{bn}=n-s]-P[T_{bn}=n-r])\right|$$
$$\le4n^{-2}ET_{0b}^2+(\max_{n/2<s\le n}P[T_{0b}=s])/P[T_{0n}=n]$$
$$+P[T_{0b}>n/2]$$
$$\le8n^{-2}ET_{0b}^2+\frac{3\varepsilon_{\{10.5(2)\}}(n/2,b)}{\theta P_\theta[0,1]},\qquad(1.1)$$

We have

$$\left|\sum_{r=0}^{[n/2]}\frac{P[T_{0b}=r]}{P[T_{0n}=n]}\right.$$
$$\times\left(\left\{\sum_{s=0}^{[n/2]}P[T_{0b}=s](P[T_{bn}=n-s]-P[T_{bn}=n-r]\right\}_+\right.$$
$$\left.\left.-\left\{\sum_{s=0}^{[n/2]}P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}P[T_{0n}=n]\right\}_+\right)\right|$$

$$\le\frac{1}{n^2P[T_{0n}=n]}\sum_{r\ge0}P[T_{0b}=r]\sum_{s\ge0}P[T_{0b}=s]\left|s-r\right|$$
$$\times\left\{\varepsilon_{\{10.14\}}(n,b)+2(r\vee s)\left|1-\theta\right|n^{-1}\left\{K_0\theta+4\phi^*_{\{10.8\}}(n)\right\}\right\}$$
$$\le\frac{6}{\theta nP_\theta[0,1]}ET_{0b}\varepsilon_{\{10.14\}}(n,b)$$
$$+4\left|1-\theta\right|n^{-2}ET_{0b}^2\left\{K_0\theta+4\phi^*_{\{10.8\}}(n)\right\}$$
$$\left(\frac{3}{\theta nP_\theta[0,1]}\right)\Big\},\qquad(1.2)$$

The approximation in (1.2) is further simplified by noting that

$$\sum_{r=0}^{[n/2]}P[T_{0b}=r]\left|\left\{\sum_{s=0}^{[n/2]}P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}\right\}_+\right.$$
$$\left.-\left\{\sum_{s=0}P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}\right\}_+\right|$$
$$\le\sum_{r=0}^{[n/2]}P[T_{0b}=r]\sum_{s>[n/2]}P[T_{0b}=s]\frac{(s-r)\left|1-\theta\right|}{n+1}$$
$$\le\left|1-\theta\right|n^{-1}E(T_{0b}1\{T_{0b}>n/2\})\le2\left|1-\theta\right|n^{-2}ET_{0b}^2,\qquad(1.3)$$

and then by observing that

$$\sum_{r>[n/2]}P[T_{0b}=r]\left\{\sum_{s\ge0}P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}\right\}$$
$$\le n^{-1}\left|1-\theta\right|(ET_{0b}P[T_{0b}>n/2]+E(T_{0b}1\{T_{0b}>n/2\}))$$
$$\le4\left|1-\theta\right|n^{-2}ET_{0b}^2\qquad(1.4)$$

Combining the contributions of (1.2) –(1.3), we thus find tha

$$\left| d_{TV}(L(\hat{C}[1,b]), L(\hat{Z}[1,b])) - (n+1)^{-1} \sum_{r\geq 0} P[T_{0b}=r] \left\{ \sum_{s\geq 0} P[T_{0b}=s](s-r)(1-\theta) \right\}_+ \right|$$

$$\leq \varepsilon_{\{7.8\}}(n,b)$$

$$= \frac{3}{\theta P_\theta[0,1]} \left\{ \varepsilon_{\{10.5(2)\}}(n/2,b) + 2n^{-1} ET_{0b} \varepsilon_{\{10.14\}}(n,b) \right\}$$

$$+ 2n^{-2} ET_{0b}^2 \left\{ 4 + 3|1-\theta| + \frac{24|1-\theta| \phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]} \right\} \qquad (1.5)$$

The quantity $\varepsilon_{\{7.8\}}(n,b)$ is seen to be of the order claimed under Conditions $(A_0),(D_1)$ and (B_{12}), provided that $S(\infty)<\infty$; this supplementary condition can be removed if $\phi^*_{\{10.8\}}(n)$ is replaced by $\phi^*_{\{10.11\}}(n)$ in the definition of $\varepsilon_{\{7.8\}}(n,b)$, has the required order without the restriction on the r_i implied by assuming that $S(\infty)<\infty$. Finally, a direct calculation now shows that

$$\sum_{r\geq 0} P[T_{0b}=r] \left\{ \sum_{s\geq 0} P[T_{0b}=s](s-r)(1-\theta) \right\}_+$$

$$= \frac{1}{2}|1-\theta| E|T_{0b} - ET_{0b}|$$

Example 1.0. Consider the point $O=(0,...,0)\in \mathbb{R}^n$. For an arbitrary vector r, the coordinates of the point $x=O+r$ are equal to the respective coordinates of the vector $r: x=(x^1,...x^n)$ and $r=(x^1,...,x^n)$. The vector r such as in the example is called the position vector or the radius vector of the point x. (Or, in greater detail: r is the radius-vector of x w.r.t an origin O). Points are frequently specified by their radius-vectors. This presupposes the choice of O as the "standard origin". Let us summarize. We have considered $\mathbb{R}^n$ and interpreted its elements in two ways: as points and as vectors. Hence we may say that we leading with the two copies of $\mathbb{R}^n$: $\mathbb{R}^n$ = {points}, $\mathbb{R}^n$ = {vectors}

Operations with vectors: multiplication by a number, addition. Operations with points and vectors: adding a vector to a point (giving a point), subtracting two points (giving a vector). $\mathbb{R}^n$ treated in this way is called an *n-dimensional affine space*. *(*An "abstract" affine space is a pair of sets , the set of points and the set of vectors so that the operations as above are defined axiomatically). Notice that vectors in an affine space are also known as "free vectors". Intuitively, they are not fixed at points and "float freely" in space. From $\mathbb{R}^n$ considered as an affine space we can precede in two opposite directions: $\mathbb{R}^n$ as an Euclidean space $\Leftarrow$ $\mathbb{R}^n$ as an affine space $\Rightarrow$ $\mathbb{R}^n$ as a manifold.Going to the left means introducing some extra structure which will make the geometry richer. Going to the right means forgetting about part of the affine structure; going further in this direction will lead us to the so-called "smooth (or differentiable) manifolds". The theory of differential forms does not require any extra geometry. So our natural direction is to the right. The Euclidean structure, however, is useful for examples and applications. So let us say a few words about it:

Remark 1.0. *Euclidean geometry.* In $\mathbb{R}^n$ considered as an affine space we can already do a good deal of geometry. For example, we can consider lines and planes, and quadric surfaces like an ellipsoid. However, we cannot discuss such things as "lengths", "angles" or "areas" and "volumes". To be able to do so, we have to introduce some more definitions, making $\mathbb{R}^n$ a Euclidean space. Namely, we define the length of a vector $a=(a^1,...,a^n)$ to be

$$|a| := \sqrt{(a^1)^2 + ... + (a^n)^2} \qquad (1)$$

After that we can also define distances between points as follows:

$$d(A,B) := |\overrightarrow{AB}| \qquad (2)$$

One can check that the distance so defined possesses natural properties that we expect: is it always non-negative and equals zero only for coinciding points; the distance from A to B is the same as that from B to A (symmetry); also, for three points, A, B and C, we have $d(A,B)\leq d(A,C)+d(C,B)$ (the "triangle inequality"). To define angles, we first introduce the scalar product of two vectors

$$(a,b) := a^1b^1 + ... + a^nb^n \qquad (3)$$

Thus $|a|=\sqrt{(a,a)}$. The scalar product is also denote by dot: $a.b=(a,b)$, and hence is often referred to as the "dot product" . Now, for nonzero vectors, we define the angle between them by the equality

$$\cos\alpha := \frac{(a,b)}{|a||b|} \qquad (4)$$

The angle itself is defined up to an integral multiple of 2π . For this definition to be consistent we have to ensure that the r.h.s. of (4) does not exceed 1 by the absolute value. This follows from the inequality

$$(a,b)^2 \leq |a|^2 |b|^2 \qquad (5)$$

known as the Cauchy–Bunyakovsky–Schwarz inequality (various combinations of these three names are applied in different books). One of the ways of proving (5) is to consider the scalar square of the linear combination $a+tb,$ where $t \in R$. As $(a+tb, a+tb) \geq 0$ is a quadratic polynomial in t which is never negative, its discriminant must be less or equal zero. Writing this explicitly yields (5). The triangle inequality for distances also follows from the inequality (5).

Example 1.1. Consider the function $f(x)=x^i$ (the i-th coordinate). The linear function dx^i (the differential of x^i) applied to an arbitrary vector h is simply h^i.From these examples follows that we can rewrite df as

$$df = \frac{\partial f}{\partial x^1} dx^1 + \ldots + \frac{\partial f}{\partial x^n} dx^n, \qquad (1)$$

which is the standard form. Once again: the partial derivatives in (1) are just the coefficients (depending on x); $dx^1, dx^2, \ldots$ are linear functions giving on an arbitrary vector h its coordinates $h^1, h^2, \ldots,$ respectively. Hence

$$df(x)(h) = \partial_{hf(x)} = \frac{\partial f}{\partial x^1} h^1 + \ldots + \frac{\partial f}{\partial x^n} h^n, \qquad (2)$$

Theorem 1.7. Suppose we have a parametrized curve $t \mapsto x(t)$ passing through $x_0 \in \mathbb{R}^n$ at $t=t_0$ and with the velocity vector $x(t_0)=\upsilon$ Then

$$\frac{df(x(t))}{dt}(t_0) = \partial_\upsilon f(x_0) = df(x_0)(\upsilon) \qquad (1)$$

Proof. Indeed, consider a small increment of the parameter $t : t_0 \mapsto t_0 + \Delta t$, Where $\Delta t \mapsto 0$. On the other hand, we have $f(x_0+h) - f(x_0) = df(x_0)(h) + \beta(h)|h|$ for an arbitrary vector h , where $\beta(h) \to 0$ when $h \to 0$. Combining it together, for the increment of $f(x(t))$ we obtain

$$\begin{aligned} &f(x(t_0+\Delta t) - f(x_0) \\ &= df(x_0)(\upsilon.\Delta t + \alpha(\Delta t)\Delta t) \\ &+\beta(\upsilon.\Delta t + \alpha(\Delta t)\Delta t).|\upsilon\Delta t + \alpha(\Delta t)\Delta t| \\ &= df(x_0)(\upsilon).\Delta t + \gamma(\Delta t)\Delta t \end{aligned}$$

For a certain $\gamma(\Delta t)$ such that $\gamma(\Delta t) \to 0$ when $\Delta t \to 0$ (we used the linearity of $df(x_0)$). By the definition, this means that the derivative of $f(x(t))$ at $t=t_0$ is exactly $df(x_0)(\upsilon)$. The statement of the theorem can be expressed by a simple formula:

$$\frac{df(x(t))}{dt} = \frac{\partial f}{\partial x^1} x^1 + \ldots + \frac{\partial f}{\partial x^n} x^n \qquad (2)$$

To calculate the value Of df at a point x_0 on a given vector υ one can take an arbitrary curve passing Through x_0 at t_0 with υ as the velocity vector at t_0 and calculate the usual derivative of $f(x(t))$ at $t=t_0$.

Theorem 1.8. For functions $f, g : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$,

$$d(f+g) = df + dg \qquad (1)$$

$$d(fg) = df.g + f.dg \qquad (2)$$

Proof. Consider an arbitrary point x_0 and an arbitrary vector υ stretching from it. Let a curve $x(t)$ be such that $x(t_0)=x_0$ and $x(t_0)=\upsilon$. Hence

$$d(f+g)(x_0)(\upsilon) = \frac{d}{dt}(f(x(t)) + g(x(t)))$$

at $t=t_0$ and

$$d(fg)(x_0)(\upsilon) = \frac{d}{dt}(f(x(t))g(x(t)))$$

at $t=t_0$ Formulae (1) and (2) then immediately follow from the corresponding formulae for the usual derivative Now, almost without change the theory generalizes to functions taking values in $\mathbb{R}^m$ instead of $\mathbb{R}$. The only difference is that now the differential of a map $F : U \to \mathbb{R}^m$ at a point x will be a linear function taking vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$ (instead of $\mathbb{R}$) . For an arbitrary vector $h \in| \mathbb{R}^n$,

$$F(x+h) = F(x) + dF(x)(h)$$

$$+\beta(h)|h| \qquad (3)$$

Where $\beta(h)\to 0$ when $h\to 0$. We have $dF=(dF^1,...,dF^m)$ and

$$dF=\frac{\partial F}{\partial x^1}dx^1+...+\frac{\partial F}{\partial x^n}dx^n$$

$$=\begin{pmatrix}\frac{\partial F^1}{\partial x^1}\cdots\frac{\partial F^1}{\partial x^n}\\ \cdots \quad \cdots \quad \cdots\\ \frac{\partial F^m}{\partial x^1}\cdots\frac{\partial F^m}{\partial x^n}\end{pmatrix}\begin{pmatrix}dx^1\\ \cdots\\ dx^n\end{pmatrix} \qquad (4)$$

In this matrix notation we have to write vectors as vector-columns.

Theorem 1.9. For an arbitrary parametrized curve $x(t)$ in $\mathbb{R}^n$, the differential of a map $F:U\to\mathbb{R}^m$ (where $U\subset\mathbb{R}^n$) maps the velocity vector $x(t)$ to the velocity vector of the curve $F(x(t))$ in $\mathbb{R}^m$:

$$\frac{dF(x(t))}{dt}=dF(x(t))(\dot{x}(t)) \qquad (1)$$

Proof. By the definition of the velocity vector,

$$x(t+\Delta t)=x(t)+\dot{x}(t).\Delta t+\alpha(\Delta t)\Delta t \qquad (2)$$

Where $\alpha(\Delta t)\to 0$ when $\Delta t\to 0$. By the definition of the differential,

$$F(x+h)=F(x)+dF(x)(h)+\beta(h)|h \qquad (3)|$$

Where $\beta(h)\to 0$ when $h\to 0$. we obtain

$$F(x(t+\Delta t))=F(x+\underbrace{\dot{x}(t).\Delta t+\alpha(\Delta t)\Delta t)}_{h}$$

$$=F(x)+dF(x)(\dot{x}(t)\Delta t+\alpha(\Delta t)\Delta t)+$$

$$\beta(\dot{x}(t)\Delta t+\alpha(\Delta t)\Delta t).\left|\dot{x}(t)\Delta t+\alpha(\Delta t)\Delta t\right|$$

$$=F(x)+dF(x)(\dot{x}(t)\Delta t+\gamma(\Delta t)\Delta t$$

For some $\gamma(\Delta t)\to 0$ when $\Delta t\to 0$. This precisely means that $dF(x)\dot{x}(t)$ is the velocity vector of $F(x)$. As every vector attached to a point can be viewed as the velocity vector of some curve passing through this point, this theorem gives a clear geometric picture of dF as a linear map on vectors.

Theorem 1.10 Suppose we have two maps $F:U\to V$ and $G:V\to W$, where $U\subset\mathbb{R}^n, V\subset\mathbb{R}^m, W\subset\mathbb{R}^p$ (open domains). Let $F:x\mapsto y=F(x)$. Then the differential of the composite map $GoF:U\to W$ is the composition of the differentials of F and G:

$$d(GoF)(x)=dG(y)odF(x) \qquad (4)$$

Proof. We can use the description of the differential .Consider a curve $x(t)$ in $\mathbb{R}^n$ with the velocity vector $\dot{x}$. Basically, we need to know to which vector in $\mathbb{R}^p$ it is taken by $d(GoF)$. the curve $(GoF)(x(t)=G(F(x(t))$. By the same theorem, it equals the image under dG of the Anycast Flow vector to the curve $F(x(t))$ in $\mathbb{R}^m$. Applying the theorem once again, we see that the velocity vector to the curve $F(x(t))$ is the image under dF of the vector $\dot{x}(t)$. Hence $d(GoF)(\dot{x})=dG(dF(\dot{x}))$ for an arbitrary vector $\dot{x}$.

Corollary 1.0. If we denote coordinates in $\mathbb{R}^n$ by $(x^1,...,x^n)$ and in $\mathbb{R}^m$ by $(y^1,...,y^m)$, and write

$$dF=\frac{\partial F}{\partial x^1}dx^1+...+\frac{\partial F}{\partial x^n}dx^n \qquad (1)$$

$$dG=\frac{\partial G}{\partial y^1}dy^1+...+\frac{\partial G}{\partial y^n}dy^n, \qquad (2)$$

Then the chain rule can be expressed as follows:

$$d(GoF)=\frac{\partial G}{\partial y^1}dF^1+...+\frac{\partial G}{\partial y^m}dF^m, \qquad (3)$$

Where dF^i are taken from (1). In other words, to get $d(GoF)$ we have to substitute into (2) the expression for $dy^i=dF^i$ from (3). This can also be expressed by the following matrix formula:

$$d(GoF)=\begin{pmatrix}\frac{\partial G^1}{\partial y^1}\cdots\frac{\partial G^1}{\partial y^m}\\ \cdots \quad \cdots \quad \cdots\\ \frac{\partial G^p}{\partial y^1}\cdots\frac{\partial G^p}{\partial y^m}\end{pmatrix}\begin{pmatrix}\frac{\partial F^1}{\partial x^1}\cdots\frac{\partial F^1}{\partial x^n}\\ \cdots \quad \cdots \quad \cdots\\ \frac{\partial F^m}{\partial x^1}\cdots\frac{\partial F^m}{\partial x^n}\end{pmatrix}\begin{pmatrix}dx^1\\ \cdots\\ dx^n\end{pmatrix} \qquad (4)$$

i.e., if dG and dF are expressed by matrices of partial derivatives, then $d(GoF)$ is expressed by the product of these matrices. This is often written as

$$\begin{pmatrix} \frac{\partial z^1}{\partial x^1} \cdots \frac{\partial z^1}{\partial x^n} \\ \dots \quad \dots \quad \dots \\ \frac{\partial z^p}{\partial x^1} \cdots \frac{\partial z^p}{\partial x^n} \end{pmatrix} = \begin{pmatrix} \frac{\partial z^1}{\partial y^1} \cdots \frac{\partial z^1}{\partial y^m} \\ \dots \quad \dots \quad \dots \\ \frac{\partial z^p}{\partial y^1} \cdots \frac{\partial z^p}{\partial y^m} \end{pmatrix} \begin{pmatrix} \frac{\partial y^1}{\partial x^1} \cdots \frac{\partial y^1}{\partial x^n} \\ \dots \quad \dots \quad \dots \\ \frac{\partial y^m}{\partial x^1} \cdots \frac{\partial y^m}{\partial x^n} \end{pmatrix}, \qquad (5)$$

Or

$$\frac{\partial z^\mu}{\partial x^a} = \sum_{i=1}^{m} \frac{\partial z^\mu}{\partial y^i} \frac{\partial y^i}{\partial x^a}, \qquad (6)$$

Where it is assumed that the dependence of $y \in \mathbb{R}^m$ on $x \in \mathbb{R}^n$ is given by the map F, the dependence of $z \in \mathbb{R}^p$ on $y \in \mathbb{R}^m$ is given by the map G, and the dependence of $z \in \mathbb{R}^p$ on $x \in \mathbb{R}^n$ is given by the composition GoF.

Definition 1.6. Consider an open domain $U \subset \mathbb{R}^n$. Consider also another copy of $\mathbb{R}^n$, denoted for distinction $\mathbb{R}^n_y$, with the standard coordinates $(y^1 \dots y^n)$. A system of coordinates in the open domain U is given by a map $F: V \to U$, where $V \subset \mathbb{R}^n_y$ is an open domain of $\mathbb{R}^n_y$, such that the following three conditions are satisfied :

(1) F is smooth;
(2) F is invertible;
(3) $F^{-1}: U \to V$ is also smooth

The coordinates of a point $x \in U$ in this system are the standard coordinates of $F^{-1}(x) \in \mathbb{R}^n_y$

In other words,

$$F: (y^1 \dots, y^n) \mapsto x = x(y^1 \dots, y^n) \qquad (1)$$

Here the variables $(y^1 \dots, y^n)$ are the "new" coordinates of the point x

Example 1.2. Consider a curve in $\mathbb{R}^2$ specified in polar coordinates as

$$x(t): r = r(t), \varphi = \varphi(t) \qquad (1)$$

We can simply use the chain rule. The map $t \mapsto x(t)$ can be considered as the composition of the maps $t \mapsto (r(t), \varphi(t)), (r, \varphi) \mapsto x(r, \varphi)$. Then, by the chain rule, we have

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \varphi}\frac{d\varphi}{dt} = \frac{\partial x}{\partial r}\dot{r} + \frac{\partial x}{\partial \varphi}\dot{\varphi} \qquad (2)$$

Here $\dot{r}$ and $\dot{\varphi}$ are scalar coefficients depending on t, whence the partial derivatives $\partial x/\partial r, \partial x/\partial \varphi$ are vectors depending on point in $\mathbb{R}^2$. We can compare this with the formula in the "standard" coordinates: $\dot{x} = e_1 \dot{x} + e_2 \dot{y}$. Consider the vectors $\partial x/\partial r, \partial x/\partial \varphi$. Explicitly we have

$$\frac{\partial x}{\partial r} = (\cos\varphi, \sin\varphi) \qquad (3)$$

$$\frac{\partial x}{\partial \varphi} = (-r\sin\varphi, r\cos\varphi) \qquad (4)$$

From where it follows that these vectors make a basis at all points except for the origin (where $r = 0$). It is instructive to sketch a picture, drawing vectors corresponding to a point as starting from that point. Notice that $\partial x/\partial r, \partial x/\partial \varphi$ are, respectively, the velocity vectors for the curves $r \mapsto x(r, \varphi)$ $(\varphi = \varphi_0 \ fixed)$ and $\varphi \mapsto x(r, \varphi)$ $(r = r_0 \ fixed)$. We can conclude that for an arbitrary curve given in polar coordinates the velocity vector will have components $(\dot{r}, \dot{\varphi})$ if as a basis we take $e_r := \partial x/\partial r, e_\varphi := \partial x/\partial \varphi$:

$$\dot{x} = e_r \dot{r} + e_\varphi \dot{\varphi} \qquad (5)$$

A characteristic feature of the basis e_r, e_φ is that it is not "constant" but depends on point. Vectors "stuck to points" when we consider curvilinear coordinates.

Proposition 1.3. The velocity vector has the same appearance in all coordinate systems.

Proof. Follows directly from the chain rule and the transformation law for the basis e_i. In particular, the elements of the basis $e_i = \partial x/\partial x^i$ (originally, a formal notation) can be understood directly as the velocity vectors of the coordinate lines

$x^i \mapsto x(x^1,...,x^n)$ (all coordinates but x^i are fixed). Since we now know how to handle velocities in arbitrary coordinates, the best way to treat the differential of a map $F:\mathbb{R}^n \to \mathbb{R}^m$ is by its action on the velocity vectors. By definition, we set

$$dF(x_0):\frac{dx(t)}{dt}(t_0) \mapsto \frac{dF(x(t))}{dt}(t_0) \qquad (1)$$

Now $dF(x_0)$ is a linear map that takes vectors attached to a point $x_0 \in \mathbb{R}^n$ to vectors attached to the point $F(x) \in \mathbb{R}^m$

$$dF = \frac{\partial F}{\partial x^1}dx^1 + ... + \frac{\partial F}{\partial x^n}dx^n$$

$$(e_1,...,e_m)\begin{pmatrix} \frac{\partial F^1}{\partial x^1} \cdots \frac{\partial F^1}{\partial x^n} \\ \cdots \ \cdots \ \cdots \\ \frac{\partial F^m}{\partial x^1} \cdots \frac{\partial F^m}{\partial x^n} \end{pmatrix}\begin{pmatrix} dx^1 \\ \cdots \\ dx^n \end{pmatrix}, \qquad (2)$$

In particular, for the differential of a function we always have

$$df = \frac{\partial f}{\partial x^1}dx^1 + ... + \frac{\partial f}{\partial x^n}dx^n, \qquad (3)$$

Where x^i are arbitrary coordinates. The form of the differential does not change when we perform a change of coordinates.

Example 1.3 Consider a 1-form in $\mathbb{R}^2$ given in the standard coordinates:

$A = -ydx + xdy$ In the polar coordinates we will have $x = r\cos\varphi, y = r\sin\varphi$, hence

$$dx = \cos\varphi dr - r\sin\varphi d\varphi$$

$$dy = \sin\varphi dr + r\cos\varphi d\varphi$$

Substituting into A, we get

$$A = -r\sin\varphi(\cos\varphi dr - r\sin\varphi d\varphi)$$
$$+r\cos\varphi(\sin\varphi dr + r\cos\varphi d\varphi)$$
$$= r^2(\sin^2\varphi + \cos^2\varphi)d\varphi = r^2 d\varphi$$

Hence $A = r^2 d\varphi$ is the formula for A in the polar coordinates. In particular, we see that this is again a 1-form, a linear combination of the differentials of coordinates with functions as coefficients. Secondly, in a more conceptual way, we can define a 1-form in a domain U as a linear function on vectors at every point of U :

$$\omega(\upsilon) = \omega_1\upsilon^1 + ... + \omega_n\upsilon^n, \qquad (1)$$

If $\upsilon = \sum e_i\upsilon^i$, where $e_i = \partial x / \partial x^i$. Recall that the differentials of functions were defined as linear functions on vectors (at every point), and

$$dx^i(e_j) = dx^i\left(\frac{\partial x}{\partial x^j}\right) = \delta^i_j \qquad (2)$$

at every point x.

Theorem 1.9. For arbitrary 1-form ω and path γ, the integral $\int_\gamma \omega$ does not change if we change parametrization of γ provide the orientation remains the same.

Proof: Consider $\left\langle \omega(x(t)), \frac{dx}{dt'} \right\rangle$ and $\left\langle \omega(x(t(t'))), \frac{dx}{dt'} \right\rangle$ As

$$\left\langle \omega(x(t(t'))), \frac{dx}{dt'} \right\rangle = \left|\left\langle \omega(x(t(t'))), \frac{dx}{dt'} \right\rangle\right| \cdot \frac{dt}{dt'},$$

Let p be a rational prime and let $K = \mathbb{Q}(\zeta_p)$. We write ζ for ζ_p or this section. Recall that K has degree $\varphi(p) = p-1$ over $\mathbb{Q}$. We wish to show that $O_K = \mathbb{Z}[\zeta]$. Note that ζ is a root of $x^p - 1$, and thus is an algebraic integer; since O_K is a ring we have that $\mathbb{Z}[\zeta] \subseteq O_K$. We give a proof without assuming unique factorization of ideals. We begin with some norm and trace computations. Let j be an integer. If j is not divisible by p, then ζ^j is a primitive p^{th} root of unity, and thus its conjugates are $\zeta, \zeta^2, ..., \zeta^{p-1}$. Therefore

$$Tr_{K/\mathbb{Q}}(\zeta^j) = \zeta + \zeta^2 + ... + \zeta^{p-1} = \Phi_p(\zeta) - 1 = -1$$

If p does divide j, then $\zeta^j = 1$, so it has only the one conjugate 1, and $Tr_{K/\mathbb{Q}}(\zeta^j) = p-1$ By linearity of the trace, we find that

$$Tr_{K/\mathbb{Q}}(1-\zeta) = Tr_{K/\mathbb{Q}}(1-\zeta^2) = ...$$
$$= Tr_{K/\mathbb{Q}}(1-\zeta^{p-1}) = p$$

We also need to compute the norm of $1-\zeta$. For this, we use the factorization

$$x^{p-1}+x^{p-2}+\ldots+1=\Phi_p(x)$$
$$=(x-\zeta)(x-\zeta^2)\ldots(x-\zeta^{p-1});$$

Plugging in $x=1$ shows that

$$p=(1-\zeta)(1-\zeta^2)\ldots(1-\zeta^{p-1})$$

Since the $(1-\zeta^j)$ are the conjugates of $(1-\zeta)$, this shows that $N_{K/\mathbb{Q}}(1-\zeta)=p$ The key result for determining the ring of integers O_K is the following.

LEMMA 1.9

$$(1-\zeta)O_K\cap\mathbb{Z}=p\mathbb{Z}$$

Proof. We saw above that p is a multiple of $(1-\zeta)$ in O_K, so the inclusion $(1-\zeta)O_K\cap\mathbb{Z}\supseteq p\mathbb{Z}$ is immediate. Suppose now that the inclusion is strict. Since $(1-\zeta)O_K\cap\mathbb{Z}$ is an ideal of $\mathbb{Z}$ containing $p\mathbb{Z}$ and $p\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$, we must have $(1-\zeta)O_K\cap\mathbb{Z}=\mathbb{Z}$ Thus we can write

$$1=\alpha(1-\zeta)$$

For some $\alpha\in O_K$. That is, $1-\zeta$ is a unit in O_K.

COROLLARY 1.1 For any $\alpha\in O_K$, $Tr_{K/\mathbb{Q}}((1-\zeta)\alpha)\in p\mathbb{Z}$

PROOF. We have

$$Tr_{K/\mathbb{Q}}((1-\zeta)\alpha)=\sigma_1((1-\zeta)\alpha)+\ldots+\sigma_{p-1}((1-\zeta)\alpha)$$
$$=\sigma_1(1-\zeta)\sigma_1(\alpha)+\ldots+\sigma_{p-1}(1-\zeta)\sigma_{p-1}(\alpha)$$
$$=(1-\zeta)\sigma_1(\alpha)+\ldots+(1-\zeta^{p-1})\sigma_{p-1}(\alpha)$$

Where the σ_i are the complex embeddings of K (which we are really viewing as automorphisms of K) with the usual ordering. Furthermore, $1-\zeta^j$ is a multiple of $1-\zeta$ in O_K for every $j\neq 0$. Thus

$Tr_{K/\mathbb{Q}}(\alpha(1-\zeta))\in(1-\zeta)O_K$ Since the trace is also a rational integer.

PROPOSITION 1.4 Let p be a prime number and let $K=|\mathbb{Q}(\zeta_p)$ be the p^{th} cyclotomic field. Then $O_K=\mathbb{Z}[\zeta_p]\cong\mathbb{Z}[x]/(\Phi_p(x))$; Thus $1,\zeta_p,\ldots,\zeta_p^{p-2}$ is an integral basis for O_K.

PROOF. Let $\alpha\in O_K$ and write

$\alpha=a_0+a_1\zeta+\ldots+a_{p-2}\zeta^{p-2}$ With $a_i\in\mathbb{Q}$. Then

$$\alpha(1-\zeta)=a_0(1-\zeta)+a_1(\zeta-\zeta^2)+\ldots$$
$$+a_{p-2}(\zeta^{p-2}-\zeta^{p-1})$$

By the linearity of the trace and our above calculations we find that $Tr_{K/\mathbb{Q}}(\alpha(1-\zeta))=pa_0$ We also have

$Tr_{K/\mathbb{Q}}(\alpha(1-\zeta))\in p\mathbb{Z}$, so $a_0\in\mathbb{Z}$ Next consider the algebraic integer

$(\alpha-a_0)\zeta^{-1}=a_1+a_2\zeta+\ldots+a_{p-2}\zeta^{p-3}$; This is an algebraic integer since $\zeta^{-1}=\zeta^{p-1}$ is. The same argument as above shows that $a_1\in\mathbb{Z}$, and continuing in this way we find that all of the a_i are in $\mathbb{Z}$. This completes the proof.

Example 1.4 Let $K=\mathbb{Q}$, then the local ring $\mathbb{Z}_{(p)}$ is simply the subring of $\mathbb{Q}$ of rational numbers with denominator relatively prime to p. Note that this ring $\mathbb{Z}_{(p)}$ is not the ring $\mathbb{Z}_p$ of p-adic integers; to get $\mathbb{Z}_p$ one must complete $\mathbb{Z}_{(p)}$. The usefulness of $O_{K,p}$ comes from the fact that it has a particularly simple ideal structure. Let a be any proper ideal of $O_{K,p}$ and consider the ideal $a\cap O_K$ of O_K. We claim that $a=(a\cap O_K)O_{K,p}$; That is, that a is generated by the elements of a in $a\cap O_K$. It is clear from the definition of an ideal that $a\supseteq(a\cap O_K)O_{K,p}$. To prove the other inclusion, let α be any element of a. Then we can write $\alpha=\beta/\gamma$ where $\beta\in O_K$ and $\gamma\notin p$. In particular, $\beta\in a$ (since $\beta/\gamma\in a$ and a is an ideal), so $\beta\in O_K$ and $\gamma\notin p$. so $\beta\in a\cap O_K$. Since $1/\gamma\in O_{K,p}$, this implies that $\alpha=\beta/\gamma\in(a\cap O_K)O_{K,p}$, as claimed.We can use this fact to determine all of the ideals of $O_{K,p}$. Let a be any ideal of $O_{K,p}$ and consider the ideal factorization of $a\cap O_K$ in O_K. write it as $a\cap O_K=p^n b$ For some n and some ideal b, relatively prime to p. we claim first that $bO_{K,p}=O_{K,p}$. We now find that

$$a=(a\cap O_K)O_{K,p}=p^n bO_{K,p}=p^n O_{K,p}$$

Since $bO_{K,p}$. Thus every ideal of $O_{K,p}$ has the form $p^n O_{K,p}$ for some n; it follows immediately that $O_{K,p}$ is noetherian. It is also now clear that $p^n O_{K,p}$ is the unique non-zero prime ideal in $O_{K,p}$. Furthermore, the inclusion $O_K \mapsto O_{K,p} / pO_{K,p}$ Since $pO_{K,p} \cap O_K = p$, this map is also surjection, since the residue class of $\alpha / \beta \in O_{K,p}$ (with $\alpha \in O_K$ and $\beta \notin p$) is the image of $\alpha\beta^{-1}$ in $O_{K/p}$, which makes sense since β is invertible in $O_{K/p}$. Thus the map is an isomorphism. In particular, it is now abundantly clear that every non-zero prime ideal of $O_{K,p}$ is maximal. To show that $O_{K,p}$ is a Dedekind domain, it remains to show that it is integrally closed in K. So let $\gamma \in K$ be a root of a polynomial with coefficients in $O_{K,p}$; write this polynomial as $x^m + \frac{\alpha_{m-1}}{\beta_{m-1}} x^{m-1} + \ldots + \frac{\alpha_0}{\beta_0}$ With $\alpha_i \in O_K$ and $\beta_i \in O_{K-p}$. Set $\beta = \beta_0 \beta_1 \ldots \beta_{m-1}$. Multiplying by β^m we find that $\beta\gamma$ is the root of a monic polynomial with coefficients in O_K. Thus $\beta\gamma \in O_K$; since $\beta \notin p$, we have $\beta\gamma / \beta = \gamma \in O_{K,p}$. Thus $O_{K,p}$ is integrally close in K.

COROLLARY 1.2. Let K be a number field of degree n and let α be in O_K then

$$N'_{K/\mathbb{Q}}(\alpha O_K) = \left| N_{K/\mathbb{Q}}(\alpha) \right|$$

PROOF. We assume a bit more Galois theory than usual for this proof. Assume first that $K/\mathbb{Q}$ is Galois. Let σ be an element of $Gal(K/\mathbb{Q})$. It is clear that $\sigma(O_K)/\sigma(\alpha) \cong O_{K/\alpha}$; since $\sigma(O_K) = O_K$, this shows that $N'_{K/\mathbb{Q}}(\sigma(\alpha)O_K) = N'_{K/\mathbb{Q}}(\alpha O_K)$. Taking the product over all $\sigma \in Gal(K/\mathbb{Q})$, we have $N'_{K/\mathbb{Q}}(N_{K/\mathbb{Q}}(\alpha)O_K) = N'_{K/\mathbb{Q}}(\alpha O_K)^n$ Since $N_{K/\mathbb{Q}}(\alpha)$ is a rational integer and O_K is a free$\mathbb{Z}$-module of rank n,

$O_K / N_{K/\mathbb{Q}}(\alpha) O_K$ Will have order $N_{K/\mathbb{Q}}(\alpha)^n$; therefore

$$N'_{K/\mathbb{Q}}(N_{K/\mathbb{Q}}(\alpha)O_K) = N_{K/\mathbb{Q}}(\alpha O_K)^n$$

This completes the proof. In the general case, let L be the Galois closure of K and set $[L:K] = m$.

IV. SENSORY SYSTEM AND LOW-LEVEL CONTROL

Polychaete and oligochaete have developed a rich variety of sensors well adapted to their lifestyle and habitat. Polychaete are highly touch sensitive; touch receptors are distributed over much of their body surface, in particular on the head and in parts of the parapodia; setae also function as such receptors. These touch receptors are used in the worm's interaction with its immediate surroundings. Many polychaete possess photoreceptors of varying complexity, ranging from simple pit eye-spots to eyes with sophisticated lenses and compound eyes. Annelids are one of the six phyla (out of the 33 metazoan phyla), where a competent optical system has evolved. Eyes usually occur in pairs on the dorsal surface of the head [10]. Nearly all polychaete possess chemoreceptors, which are specialized cells, sensitive to chemicals dissolved in the environment. They are scattered over much of their body surface. Statocysts also occur, mainly in burrowing and tube dwelling polychaete; they function as georeceptors and help the animal maintain proper orientation in its burrow. The sensor integration into the robotic platform is still in progress; many issues have to be addressed both from the theoretical and from the technological point of view in order to obtain a perceptive and reactive behavior. A possible concept of the devised final biomimetic system is illustrated in Figure 6. The current activity on this topic is twofold. From the technological point of view, the authors are going to integrate simple contact sensors (PVDF foil) into the silicone skin of the artificial earthworm illustrated in Figure 3. The PVDF sensor should distinguish the sinusoidal deformation of the shell from any external contacts produced by obstacles on the pattern (Figure 7). This signal could be used for the reactive control at the module level. Concerning the overall architecture, the authors are investigating a control structure allowing the integration of up to 64 locomotion units (such as the one illustrated in Figure 6), each one embedding a microcontroller, the module actuator, and two contact sensors. One of these modules (i.e., the head module or master module)
would consist of the same hardware but of a different controller able to generate the gait pattern for the overall prototype (i.e., for the slave modules). This architecture would allow overcoming the wiring problems that pose dramatic limitations to the current prototypes.

V. THE DEVELOPMENT OF A CYBERNETIC HAND PROSTHESIS

Imitating the capabilities of the human manipulation systems has been the dream of scientists and engineers for centuries. In fact, developing a truly humanlike artificial hand is probably one of the most widely known paradigms of bionics [11]. The

IST-FET CYBERHAND Project aims at combining recent advances in various fields of neuroscience, medicine, and technology in order to investigate the concept of a truly bionic hand, i.e., an artificial hand whose shape, functions, and, above all, perception, are so advanced that it is interchangeable with the natural hand. Within the framework of CYBERHAND, the consortium aims to achieve a number of important and tangible results, both in terms of enhanced basic knowledge and in terms of technological implementations. The main result of the CYBERHAND Project will be the development of a new kind of hand prosthesis (i.e., a cybernetic prosthesis) able to recreate the natural link that exists between the hand and the central nervous system (CNS) by exploiting the potentialities of implantable interfaces with the peripheral nervous system. The natural hand is controlled by using the neural commands (i.e., the efferent neural signals) going from the CNS to the peripheral nervous system (to recruit the different muscles). At the same time, the information (concerning the position of the fingers, the force produced during grasp, and the slippage of the objects) obtained from the natural sensors (mechanoreceptors and muscle spindles) are brought to the CNS by activation of the afferent peripheral nerves. The hand prosthesis, which is being developed in the framework of the CYBERHAND Project, aims to implement the above described structure (see Figure 8). The possibility of restoring the perception of the natural hand (i.e., the possibility of delivering a natural sensory feedback to the user) will increase the acceptability of the artificial device. This result will be achieved also by developing biomimetic sensors replicating the natural sensors of the hand, and specific neural electrodes will allow selective stimulation (to deliver the sensory feedback) and recording (to extract the user's intentions). The following presents, in brief, the main results and the current works on the neural interfaces and the biomechatronic hand topics.

VI. RESULTS AND CURRENT WORK ON NONINVASIVE AND INVASIVE NEURAL INTERFACES

Two different steps are under investigation: 1) the development of a noninvasive neuromechatronic interface based on the processing of the electromyographic (EMG) signal and on the use of external systems to deliver a cognitive feedback to the user and 2) the development of an invasive neuromechatronic interface, achieved by using different kinds of electrodes. The noninvasive approach is being used for investigating the possibility of controlling the multiple degrees of freedom (DOF) prosthesis by processing the EMG signals [12]. The EMG signal is a simple and easy to obtain source of information on what the user of a prosthesis would like to do with her/his artificial hand. Surface electrodes are easy to use and manage, and they do not require any surgery. Moreover, there is no harness that could limit the movements of the forearm. It is possible to control an active device with just one electrode placed on the residual limb. However, it is important to point out that among the different robotic artifacts (exoskeletons, teleoperated robots, etc.), the EMG-based control of hand prostheses is the more challenging. In fact, in this case it is not possible to use the "homologous" natural muscles to control the artificial movements of the device. For example, the muscles related to the movements of the wrist (extensor and flexor carpi) are in many cases no longer available because of the amputation; for this reason, it is necessary to code this movement with other voluntary movements (e.g., the flexion/ extension of the elbow). This situation asks for the development of a complex approach using advanced pattern recognition techniques [12]. During the first phase of the project, the EMG signals from the biceps and triceps of five able-bodied subjects (participating after providing informed consent) have been used to discriminate two different upper-limb movements (elbow flexion-extension and forearm pronation-supination) in order to allow the subjects to control two different kinds of prehension (palmar and lateral grasps [13]). Three levels of force for each grasp were coded. The level of force was extracted from the "intensity" of the activation of the muscles. An algorithm based on statistical (the "generalized likelihood ratio" (GLR) test [14]) and neurofuzzy algorithms has been used. The rate of successful classification was around 95% with the neurofuzzy classifier. This latter performance is quite similar to the state of the art in this field (see [13] and [15] for examples), even if in our case fewer electrodes have been used. Moreover, it is important to point out that this is one of the few examples in literature of the real-time control of a multi- DOF prosthetic hand. In fact, in many cases the papers address the problems of the EMG-based control of a prosthesis "simulating" the existence of the robotic device. After this preliminary phase, the possibility of controlling more DOF is currently under investigation. Ten able-bodied subjects have been enrolled in the experiments. During this phase, the EMG activity of several muscles from the shoulder and the upper arm will be recorded (e.g., deltoid, trapetius, biceps, triceps). The movements of the shoulder will be used to select the different DOF of the hand. Jerky elevation movements of the shoulder will allow locking-unlocking the grasping type selected. At the same time, the performance achieved by delivering a sensory feedback to the subject by means of several approaches that do not require the implant of electrodes (e.g., mechanical stimulation or electrical stimulation) will be analyzed in order to understand the noninvasive approach's potential and its advantages. According to the invasive approach, three different kinds of electrodes—regeneration type [16], cuff type, and longitudinal intrafascicular electrodes (LIFEs) (see Figure 9)—have been developed and implanted in different animal models (rats and rabbits). These experiments are useful for verifying the bandwidth (i.e., the amount of information that can be extracted from the signals recorded and can be delivered to the user by means of neural stimulation) of these two electrodes. In particular, for the sieve electrodes it is crucial to understand what degree of regeneration is obtainable with the new electrodes, not only from a histological point of view but also in addressing the issues related to the extraction of the information carried in the recorded signals, specifically:

- What is the degree of similarity of the recorded signals to cuff signals and to needle signals (i.e., microneurography)?
- What is the influence of the afferent signals on the efferent signals, and what kind of countermeasures can be envisaged in order to reduce the interference? Two different experimental trials are under development in order to investigate the possibility of delivering a sensory feedback and to analyze the possibility of extracting information from the neural signals recorded using the electrodes to control the biomechatronic prosthesis. The experiments on sensory feedback will be carried out in three different phases (in collaboration with the Consortium of the IST-FET ROSANA Project).
- Phase 1: Development of a model to correlate the sensory stimuli delivered to the hindpaw of the rat to the signals recorded from the holes of the sieve electrodes.
- Phase 2: Update of the model developed during Phase 1 by comparing the cortical signals obtained during sensory and electrically induced stimulation.
- Phase 3: Deliver the sensory feedback by using the signals recorded from the artificial sensors developed in the framework of the project.

Efferent signals are commonly elicited in the animal model, provoking some pain in the limb (for example by means of a laser-assisted stimulation). This strategy may have some drawbacks; for example, it could be not easy to obtain a repeatable movement (and thus repeatable efferent signal). Two other protocols are under investigation: 1) a protocol based on the reward concept (the rat moves the hind limb in a specific direction and obtains some food; moving the limb in another directions brings it some water) and 2) a protocol based on the reaction of the rats to movements imposed on its limb. If the experimenter pushes or pulls the limb in a specific direction, the rat is supposed to react, bringing the limb to a rest position and thereby activating different muscles for each specific direction.

VII. RESULTS AND CURRENT WORK ON BIOMECHATRONIC ARTIFICIAL HAND

A new biomechatronic artificial hand is under development. It will interact with the environment according to the patient's intentions and will generate the sensory information for the low-level control and the cognitive feedback systems. A three dimensional (3-D) CAD model of the hand has been created using ProMechanica Motion. This model is useful to evaluate the performance of every underactuated hand based on the RTR2 finger mechanism [17]. It is the result of a design process that can be applied for developing a generic robotic or prosthetic hand with a cable actuation system. The human hand geometric and kinematics characteristics have been studied in order to develop a hand much closer to the anthropomorphic size and movements. In particular, we focused on the fingers' sizes and joints range. Using the model, it has been possible to optimize the thumb position in order to mimic the human grasps and to dimension the electromagnetic motors. The resulting biomechatronic hand is depicted in Figure 10. It is able to perform several functional grasps (e.g., the lateral, the cylindrical, the pinch, and the tridigital grasps) that will be controlled by means of the neural interfaces developed in the CYBERHAND framework or by means of EMG signals (see the previous section "Results and Current Work on Noninvasive and Invasive Neural Interfaces"). The sensory system is the core of the CYBERHAND control system, and it should have a twofold function.

- It should provide input signals for the low-level control loop of the grasping phase, thus enabling local and autonomous control of the grasp without requiring the user's attention and reaction to incipient slippage. The low-level control system should increase the grasping force as soon as incipient slippage occurs and the object is going to slip, and thus it should replicate the user's natural reaction without requiring his/her attention and specific effort. The regulation grasping force must be an unconscious process, since humans don't feel muscles when they use them. The ideal control requires a simple starting command to perform a secure grasp, independently of the specific object characteristics in terms of shape, size, and texture.

◆ The artificial sensory system should generate sensory signals (contacts, slippage, hand posture, and surface texture) to be transmitted to the user through an appropriate neural interface (high-level control loop) and neural algorithms. The transmission of sensory information directly stimulating the nervous system (afferent nerves) by means of the neural interfaces will be exploited during the CYBERHAND Project.

Extensive proprioceptive and exteroceptive sensory systems have been developed in the Progressive and Adaptive Learning for Object Manipulation (PALOMA) Project framework. The CYBERHAND sensory system is the evolution of the PALOMA Hand sensory system in order to achieve the right compromise between the system complexity and the space limitations. Proprioception is included in order to provide the required information on all the phalanges of the hand. The sensory system includes:

◆ 15 Hall-effect sensors embedded in all the joints of each finger

◆ an incremental magnetic encoder and two stroke end Hall-effect sensors on each of the six motors

◆ five tension sensors on the cables.

The Hall-effect sensors and the encoders are used to provide information about the position of all the phalanges during grasping and manipulation tasks. The tensiometers that measure the tension on the cables controlling the fingers flexion are meant to mimic the function of the Golgi tendon organs that give information about the tendon stretches. The exteroceptive sensory system currently includes:

◆ flexible contact sensors

◆ three-axial strain gauge force sensors. Contact sensors provide on-off information to the tactile sensing system. Efforts have been made in order to confer a high contact sensitivity (about 10 mN) to emulate the mechanoreceptors of the human hand in an engineering implementation, according to neurophysiology studies [18]. The three-axial strain gauge sensors will be mounted on the thumb, index, and middle fingertips for detecting the three components of an applied force.

VIII. THE IST-FET PALOMA PROJECT, BIOLOGICALLY INSPIRED MULTINETWORK ARCHITECTURE

In robotics, biology has been a source of inspiration for the development of biomimetic components as well as new control models for biomorphic robotic platforms. But the advances of robotics technology in the development of humanlike components, i.e., sensors and actuators, is improving the opportunities of its application in the study of humans, as a tool for neurophysiologists, physiologists, neuroscientists, and psychologists to validate biological models and to carry out experiments that may be difficult or impossible with human beings. Therefore, the interaction between biological science and robotics becomes twofold [19], [20] (see Figure 11): on one hand, biology provides the knowledge of the human system needed to build humanoid robots (or humanlike components) [21], and, on the other hand, anthropomorphic robots represent a helpful platform for experimental validation of theories and hypotheses formulated by scientists [22]. The PALOMA (a biologically inspired multinetwork architecture) Project is aimed at developing an anthropomorphic robotic manipulation platform, which mimics human mechanisms of perception and action, and can implement neurophysiological models of sensory-motor coordination through a strict interaction between the roboticist and the neuroscientist partners. The system developed in the PALOMA Project is composed of sensors and actuators replicating some level of anthropomorphism in the physical structure and/or in the functionality. It is worth noting that their specifications are defined together by roboticists and neuroscientists: on the neuroscientific side, the sensory-motor functionality that the robotic platform should possess and, on the robotic side, the best available robotics technology.

IX. NEUROPHYSIOLOGIC REQUIREMENTS

Addressing sensory-motor coordination control schemes for grasping and manipulation, the anthropomorphic model considered as reference in PALOMA is the human upper torso, including an arm, one hand, a head, and the corresponding sensory apparatuses. The human arm can be modeled with a shoulder with at least 3 DOF, an elbow with 1 DOF, and a wrist with 3 DOF. The shoulder and elbow allow positioning the hand in the workspace, while the wrist allows defining the orientation of the hand. The biological motor control for the arm requires proprioceptive information of the arm as provided by the muscle spindles and the Golgi tendon organs. No special requirements have been formulated for the arm speed and acceleration, in consideration of the addressed task of grasping. For the hand, studies on the human hand and its functionality show that at least three fingers are necessary to perform most human grasps [23]; hence, the human model of the hand has to include at least the thumb, the index finger, and the middle finger, with 3 DOF for the index and middle fingers and 4 DOF for the thumb, and the proprioceptive and somatosensory apparatus. The main human eye movements considered are saccades (rapid eye movements that change fixation from one target to another) and smooth pursuit (slow, smooth eye movements that enable one to follow a steadily moving target). Even though eye muscles allow small rotations of the eyeballs, we can approximate the eye model with a common tilt movement and independent pan movements allowing eye vergence. To approximate the human eye range of motion, 120° should be achieved for the tilt and 60° for the pan movements. Especially for saccades, eye speed is an important requirement; in humans it is, on average, around 300°/s, but can reach as high a speed as 900°/s [24]. The intraocular distance in humans is approximately 70 mm. The accuracy of the head and eye movements should be enough to guarantee that a point of interest is never put outside the fovea by positioning errors. This depends on a number of factors, including the sensor size, the focal length, and the distance of the point of interest. By considering likely values for such parameters, a maximum error of 2° has been calculated. For the neck, the ventral/dorsal flexion, lateral flexion, and rotation are considered [25], [26]; according to [26], ventral and dorsal flexions occur around different axes; therefore, a total of 4 DOF are required. The reference ranges of motion are 45°–60° for the ventral flexion, 55°–60° for the dorsal flexion, 40° for the lateral flexion, and 70°–80° for the rotation. Regarding the perception system, two basic apparatuses were indicated as important for the addressed applications:

◆ a somesthetic perception system, providing sensory information on the actual status of arm and hand, usually related via proprioceptive and somatosensory signals. The classifications adopted in neuroscience are fast adapting (FA) and slowly adapting (SA) receptors of different types, related to their presence in the superficial (I) or deep (II) skin layers. The physiological model of the somatosensory system consists of tactile FAI, FAII, SAI, and SAII afferent signals [27], while the proprioception is analogous to the arm (i.e., information from muscle spindles and Golgi tendon organs). The artificial sensory system should provide information on contact making and breaking, slip friction between object and fingertips, object shape, and force vector at the contact points

◆ a visual apparatus, to give information about the hand configuration during the reaching and grasping task and about the object to be grasped, with specific reference to shape, orientation of grip axis with respect to gravity, size (for size-weight association), and position.

X. DEVELOPMENT OF THE PALOMA HUMANLIKE ROBOTIC PLATFORM

The overall PALOMA robotic platform (shown in Figure 12)
consists of

◆ a three-fingered robotic hand, with a somatosensory system, which includes proprioceptive and tactile systems

◆ a robotic head equipped with a stereoscopic vision system ◆ an anthropomorphic 8-DOF robot arm, with a proprioceptive sensory system. The basic characteristics of all the components are described in the following.

XI. THE ROBOTIC HAND

A three-fingered anthropomorphic hand has been developed, starting from the biomechatronic three-fingered RTR2 hand [17], [28], [29]. The PALOMA hand has been designed for grasping at least the set of experimental objects identified by the neuroscientists, for generating bioinspired sensory information, and for having anthropomorphic dimensions and weight. Other advanced technical solutions have been analyzed, but they do not respect the PALOMA size and weight requirements [30], [31]. The grasping kinematics and dynamics have been analyzed by means of a specific 3-D CAD model of the hand created using ProMechanica Motion. The number of grasping configurations has been improved by changing the thumb mechanism and position and designing a new palm (see Figure 13). Regarding the mechanical structure, four dc motors (three are extrinsic and control the flexion/extension movements of the three fingers independently, and the last one is integrated in the palm and used for the adduction/abduction movement of the thumb) have been used for actuating 10 DOF, so each finger is underactuated, and the mechanism is the same as the RTR2 and CYBERHAND hands. The total weight (including the actuators) is about 500 g. The perception system includes proprioceptive and exteroceptive sensory systems in order to achieve the neurophysiologic requirements. Each finger is provided with a number of sensory signals: 1) a high density matrix of on-off tactile sensitive areas for each finger, distributed on the volar and lateral sides of the phalanges [see Figure 14(a)]; 2) one 3-D force sensor embedded in the fingertip, providing the three force components of the contact [see Figure 14(b) and (c)]; 3) three Hall-effect sensors (one for each joint) mounted as angle joint sensors; 4) one tension sensor mounted at the base of the finger; and 5) one encoder. In addition, an accelerometer is mounted within the palm for detecting the contact of the hand with the environment. In addition to the sensors currently integrated onto the robotic hand, a micro 3-D force sensor (2300 μm × 2300 μm × 1300 μm) is under development at Scuola Superiore Sant'Anna. Together with the neuroscientific partners, the possible distribution of a number of such sensors in the artificial hand skin that is under development has been studied and defined. At present, the missing steps of the fabrication process of the sensor's back side have been successfully completed. Also, the process for the front side (the silicon substrate, i.e., support wafer, with aluminum metallizations and gold pads) has been studied, the related masks have been designed, the process has been completed, and the sides have been assembled (see Figure 15).

XII. THE ROBOTIC HEAD AND VISION SYSTEM

With respect to the vision system, in order to focus the anthropomorphism requirements, a retinalike vision system is simulated on the robotic head. It is based on space variant images whose resolution is higher in the center (fovea) and degrades towards the periphery, as an imitation of images generated onto the human retina (see Figure 16). In particular, the arrangement of pixels is based on a circular structure: a constant number of pixels (252) is arranged along 110 concentric circles, with decreasing width from the periphery to the center; in the central area of 42 rings, named fovea, the number of pixels is decreased by six in each ring. Therefore, these images have a total of 33,193 pixels: 110 rings with 252 pixels in the periphery and 42 rings with a number of pixels decreasing toward the center in the fovea, corresponding to a standard image of 1,090 × 1,090 pixels [32]. The main advantage of this kind of sensor is that the amount of data is drastically reduced and faster processing can be achieved, provided that the eyes are moved in a continuous tracking of the points of interest (it happens in humans). This allows inclusion of visual processing in the control loop of the head and eye movements. On the other hand, retinalike cameras need very fast and accurate movements in order to focus the points of interest and have good vision. Therefore, it is crucial that they are mounted on a robotic head that is able to quickly and precisely move them. The mechanical structure and the performance of the robotic head (Figure 17) have been designed based on the model and performance of the human head in terms of DOF, ranges of motion, speeds, and accelerations. The resulting head has a total of 7 DOF: 4 DOF on the neck (one yaw, two pitches at different heights, and one roll), 1 DOF for a common eye tilt movement, and 2 DOF for independent eye pan movements. The performances of the robotic head in terms of ranges of motion, speeds, and accelerations are reported in Table 3. The 2-DOF performing pan movement of the eye permits vergence of the two cameras, thus allowing foveation of targets. The performance of the head allows performance of the human eye movements of smooth pursuit and saccades, as well. The head is equipped with incremental encoders for measuring the positions of all the joints as proprioceptive information. The direction of gaze is calculated from the geometrical specifications of the robotic head and from the configuration of the eyes in the joint space. For the smooth pursuit movements, a simplified vestibulo-ocular reflex (VOR) has been implemented to counterbalance the eye movements with respect to the head movements.

XIII. THE ROBOT ARM

The robotic arm integrated in the platform is an 8-DOF robot arm named Dexter and developed by S.M. Scienzia Machinale, Pisa, Italy, as an assistive robot. Its physical structure is highly anthropomorphic (see Figure 12), with the link arrangement reproducing the human body from trunk to wrist. A trunk, a shoulder, an elbow, and a wrist can be identified in the Dexter kinematic structure and, as a consequence, human movements in the interaction with the environment can be easily mimicked. The mechanical transmission system is realized through steel cables (which allow the six distal motors to be located on the first link, representing the trunk) by achieving low weight and low inertia for the distal joints. The first two proximal joints are aimed at prepositioning the distal 6-DOF manipulator to increase the overall workspace. They also help compensate the difference in the number of DOF between the human shoulder (3 DOF) and the Dexter shoulder (2 DOF). The proprioceptive information on the position of all the joints are provided by incremental encoders located on each motor. The arm can be controlled through a standard stiff proportional-integral differential (PID) controller and through interaction controls with self-adjusting compliance [33]. Furthermore, original control schemes have been developed, aimed at implementing biomorphic control mechanisms based on the combination of a feedforward control loop with a feedback control loop [34].

XIV. CONCLUSIONS

Considering the current evolution of biomechatronics and biorobotics research towards parallel directions, privileging either the improvement of biomimetic hardware performances, or the improvement of humanlike behavior, the 5th Framework Programme of the European Commission provides the authors with the opportunity to explore innovative biorobotic systems at three different levels: a biomimetic wormlike robot to be applied in endoscopy; a highly anthropomorphic robotic hand to be used as a cybernetic prosthesis connected with the human body; and a humanlike robotic platform for manipulation, to be used for conducting experiments on sensory-motor coordination models. The ongoing experience of the BIOLOCH Project shows that a sophisticated hardware imitating the mechanisms of low-level animals can reach a high degree of effectiveness by overcoming the limitations of control problems. The main issues behind this activity are essentially related to miniaturization and merging technologies able to go beyond the limits of traditional robot design. The CYBERHAND Project explores the application of bionic systems with a moderate level of intelligence to be integrated in humans. The authors propose the development of a cybernetic hand prosthesis connected to the nervous system by means of specific interfaces implementing a lifelike perception and control of a specific biomechatronic hand prosthesis. The PALOMA Project explores the application of robotics in the study of man, by providing artificial platforms for validating neuroscientific models. The authors propose a work on the development of an anthropomorphic robotic platform for sensory-motor coordination in human grasping and inspired by the analysis of neurophysiologic specifications for the human actuators and sensory systems. The integrated platform is being used for implementing a multinetwork architecture that correlates sensory and motor signals and for validating a five-step model of progressive learning of grasping and manipulation skills [35] mimicked from human babies.

A. *Authors and Affiliations*

Dr Akash Singh is working with IBM Corporation as an IT Architect and has been designing Mission Critical System and Service Solutions; He has published papers in IEEE and other International Conferences and Journals.

He joined IBM in Jul 2003 as a IT Architect which conducts research and design of High Performance Smart Grid Services and Systems and design mission critical architecture for High Performance Computing Platform and Computational Intelligence and High Speed Communication systems. He is a member of IEEE (Institute for Electrical and Electronics Engineers), the AAAI (Association for the Advancement of Artificial Intelligence) and the AACR (American Association for Cancer Research). He is the recipient of numerous awards from World Congress in Computer Science, Computer Engineering and Applied Computing 2010, 2011, and IP Multimedia System 2008 and Billing and Roaming 2008. He is active research in the field of Artificial Intelligence and advancement in Medical Systems. He is in Industry for 18 Years where he performed various role to provide the Leadership in Information Technology and Cutting edge Technology.

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